On the Hyers-Ulam stability
of Pexider–type extension
of the Jensen-Hosszu equation

Zygfryd Kominek

Abstract
We consider the following pexiderized version of Jensen-Hosszu equation of the form
\[ 2f\left(\frac{x+y}{2}\right) = g(x+y-xy) + h(xy), \]
where \( f, g, h \) are unknown real-valued functions of a real variable. We prove that \( f, g, h \) are affine functions and, moreover, we prove that these equation is stable in the Hyers-Ulam sense.

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Functional equation of the form
\[ 2f\left(\frac{x+y}{2}\right) = f(x+y-xy) + f(xy), \quad x, y \in \mathbb{R} \]
is called Jensen-Hosszu equation. In \cite{2} we have proved that Jensen-Hosszu equation is equivalent to the Jensen equation
\[ 2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \quad x, y \in \mathbb{R}. \]
The general solution of these equations are of the form \( f(x) = a(x) + c, \quad x \in \mathbb{R}, \)
where \( a : \mathbb{R} \to \mathbb{R} \) is an additive function and \( c \) is a real constant.

\footnote{Institute of Mathematics, Silesian University, Bankowa 14, PL-40-007 Katowice, Poland, \textit{e-mail: zkominek@ux2.math.us.edu.pl}}
Let \( \delta \geq 0 \) be a fixed real number and let \( f, g, h : \mathbb{R} \to \mathbb{R} \) be functions satisfying the following condition

\[
|2f\left(\frac{x+y}{2}\right) - g(x + y - xy) - h(xy)| \leq \delta, \quad x, y \in \mathbb{R}.
\]

Putting here \( x = y = 0 \) we get

\[
|2f(0) - g(0) - h(0)| \leq \delta.
\]

If \( F(x) = f(x) - f(0), \ G(x) = g(x) - g(0), \ H(x) = h(x) - h(0), \ x \in \mathbb{R} \), then the triple \( \{ F, G, H \} \) satisfies the analogue condition, i.e.,

\[
|2F\left(\frac{x+y}{2}\right) - G(x + y - xy) - H(xy)| \leq 2\delta, \quad x, y \in \mathbb{R},
\]

and, moreover,

\[
F(0) = G(0) = H(0) = 0.
\]

Setting \( y = 0 \) in (1) we obtain

\[
|2F\left(\frac{x+0}{2}\right) - G(x)| \leq 2\delta, \quad x \in \mathbb{R}.
\]

For arbitrary \( u \in \mathbb{R} \) and \( v \leq 0 \) the equation

\[
z^2 - (u+v)z + v = 0
\]

has two solutions \( x \) and \( y \) fulfilling the following equalities

\[
u + v = x + y \quad \text{and} \quad v = xy.
\]

Consequently,

\[
|2F\left(\frac{u+v}{2}\right) - G(u) - H(v)| \leq 2\delta, \quad u \in \mathbb{R}, \ v \leq 0.
\]

Setting \( u = 0 \) in (3) we obtain

\[
|2F\left(\frac{0+v}{2}\right) - H(v)| \leq 2\delta, \quad v \leq 0.
\]

By virtue of (2), (3) and (4), for all \( u \in \mathbb{R} \) and each \( v \leq 0 \), we have

\[
|2F\left(\frac{u+v}{2}\right) - 2F\left(\frac{u}{2}\right) - 2F\left(\frac{v}{2}\right)|
\]

\[
\leq |2F\left(\frac{u+v}{2}\right) - G(u) - H(v)| + |2F\left(\frac{u}{2}\right) - G(u)| + |2F\left(\frac{v}{2}\right) - H(v)| \leq 6\delta,
\]

which can be rewritten to the following equivalent form

\[
|F(u+v) - F(u) - F(v)| \leq 3\delta, \quad u \in \mathbb{R}, \ v \leq 0.
\]

According to the well-known theorem ([I], for example) there exists a uniquely determined additive function \( A : \mathbb{R} \to \mathbb{R} \) such that

\[
|F(v) - A(v)| \leq 3\delta, \quad v \leq 0.
\]
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Using also (2) we obtain

\[(7) \quad |G(v) - A(v)| \leq |G(v) - 2F\left(\frac{v}{2}\right)| + |2F\left(\frac{v}{2}\right) - 2A\left(\frac{v}{2}\right)| \leq 8\delta, \quad v \leq 0,\]

and, similarly, using (4) instead of (2)

\[(8) \quad |H(v) - A(v)| \leq 8\delta, \quad v \leq 0.\]

It follows from (3) (by putting \(u = -v\)) that

\[|G(-v) + H(v)| \leq 2\delta, \quad v \leq 0.\]

For arbitrary \(v > 0\) we have

\[|G(v) - A(v)| \leq |G(v) + H(-v)| + |A(-v) - H(-v)| \leq 2\delta + 8\delta = 10\delta,\]

which together with (7) implies that

\[(9) \quad |G(u) - A(u)| \leq 10\delta, \quad u \in \mathbb{R}.\]

According to (9) and (2)

\[|F(u) - A(u)| \leq \frac{1}{2}|2F(u) - G(2u)| + \frac{1}{2}|G(2u) - A(2u)| \leq 6\delta, \quad u \in \mathbb{R}.\]

Putting \(y = 1\) and \(x = v > 0\) in (1) we get

\[|2F\left(\frac{v+1}{2}\right) - G(1) - H(v)| \leq 2\delta,\]

and, consequently,

\[|H(v) - A(v)| \leq |H(v) + G(1) - 2F\left(\frac{v+1}{2}\right)| + 2|F\left(\frac{v+1}{2}\right) - A\left(\frac{v+1}{2}\right)| + |G(1) - A(1)| \leq 24\delta.\]

Therefore,

\[|F(x) - A(x)| \leq 6\delta, \quad |G(x) - A(x)| \leq 10\delta \quad \text{and} \quad |H(x) - A(x)| \leq 24\delta, \quad x \in \mathbb{R}.\]

Now, we are in a position to formulate our main result.

**Theorem 0.1** Let \(\delta \geq 0\) be a fixed real number and let \(f, g, h : \mathbb{R} \to \mathbb{R}\) be functions satisfying the following condition

\[|2f\left(\frac{x+y}{2}\right) - g(x+y-xy) - h(xy)| \leq \delta, \quad x, y \in \mathbb{R}.\]

Then there exist functions \(f_1, g_1, h_1 : \mathbb{R} \to \mathbb{R}\) fulfilling the equation

\[2f_1\left(\frac{x+y}{2}\right) - g_1(x+y-xy) - h_1(xy) = 0, \quad x, y \in \mathbb{R}\]

and the following estimations

\[|f(x) - f_1(x)| \leq 7\delta, \quad |g(x) - g_1(x)| \leq 11\delta, \quad \text{and} \quad |h(x) - h_1(x)| \leq 24\delta, \quad x \in \mathbb{R}.\]
**Proof.** Let $F, G, H$ and $A$ have the same meaning as above and let $d = 2f(0) - g(0) - h(0)$. As we observed $|d| \leq \delta$. We define functions $f_1, g_1$ and $h_1$ by the formulas

$$f_1(x) = A(x) + f(0) - d, \quad g_1(x) = A(x) + g(0) - d, \quad h_1(x) = A(x) + h(0), \quad x \in \mathbb{R}.$$ 

Then

$$2f_1\left(\frac{x + y}{2}\right) - g_1(x + y - xy) - h_1(xy) = 2f(0) - g(0) - h(0) - d = 0,$$

and

$$|f(x) - f_1(x)| = |f(x) - f(0) + d - A(x)| \leq |F(x) - A(x)| + |d| \leq 7\delta, \quad x \in \mathbb{R},$$

$$|g(x) - g_1(x)| \leq |G(x) - A(x)| + |d| \leq 11\delta,$$

$$|h(x) - h_1(x)| = |H(x) - A(x)| \leq 24\delta,$$

as required. \hfill \Box

**Remark 0.1** The assertion of Theorem 1 says, in another words, that pexiderized Jensen-Hosszú equation of the form

$$2f\left(\frac{x + y}{2}\right) = g(x + y - xy) + h(xy), \quad x, y \in \mathbb{R},$$

is stable in the Hyers-Ulam sense.

If we take $\delta = 0$ we obtain the solution of the pexiderized Jensen-Hosszú equation.

**Corollary 0.1** Functions $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy pexiderized Jensen-Hosszú equation if and only if there exist an additive function $A : \mathbb{R} \to \mathbb{R}$ and real constants $a$ and $b$ such that

$$(10) \quad f(x) = A(x) + a, \quad g(x) = A(x) + b, \quad h(x) = A(x) + 2a - b, \quad x \in \mathbb{R}.$$ 

**Proof.** Putting $\delta = 0$, from the proof of our Theorem, we get

$$f(x) = A(x) + f(0) - d, \quad g(x) = A(x) + g(0) - d, \quad h(x) = A(x) + h(0), \quad x \in \mathbb{R},$$

where $d = 2f(0) - g(0) - h(0)$. If $a = f(0) - d$, $b = g(0) - d$ we obtain hence (10). On the other hand, functions defined by (10) fulfilled the equation

$$2f\left(\frac{x + y}{2}\right) = g(x + y - xy) + h(xy), \quad x, y \in \mathbb{R}.$$ 

\hfill \Box

**Remark 0.2** Our main result generalizes an earlier author’s result on the Hyers-Ulam stability of the Jensen-Hosszú equation [2].
References


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