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On the Hyers-Ulam stability of Pexider– type extension of the Jensen-Hosszu equation

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Abstract

We consider the following pexiderized version of Jensen-Hosszú equation of the form

$$2f(\frac{x+y}{2}) = g(x+y-xy) + h(xy),$$

where f, g, h are unknown real-valued functions of a real variable. We prove that f, g, h are affine functions and, moreover, we prove that these equation is stable in the Hyers-Ulam sense.

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Functional equation of the form

$$2f(\frac{x+y}{2}) = f(x+y-xy) + f(xy), \qquad x, y \in \mathbb{R}$$

is called Jensen-Hosszú equation. In [2] we have proved that Jensen-Hosszú equation is equivalent to the Jensen equation

$$2f(\frac{x+y}{2}) = f(x) + f(y), \qquad x, y \in \mathbb{R}.$$

The general solution of these equations are of the form $f(x) = a(x) + c,\ x \in \mathbb{R}$, where $a : \mathbb{R} \to \mathbb{R}$ is an additive function and c is a real constant.

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Let $\delta \geq 0$ be a fixed real number and let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions satisfying the following condition

$$|2f(\frac{x+y}{2}) - g(x+y-xy) - h(xy)| \leq \delta, \qquad x, y \in \mathbb{R}.$$

Putting here $x = y = 0$ we get

$$|2f(0) - g(0) - h(0)| \leq \delta.$$

If $F(x) = f(x) - f(0),\ G(x) = g(x) - g(0),\ H(x) = h(x) - h(0),\ x \in \mathbb{R}$, then the triple $\{F, G, H\}$ satisfies the analogue condition, i.e.,

$$(1) \qquad |2F(\frac{x+y}{2}) - G(x+y-xy) - H(xy)| \leq 2\delta, \qquad x, y \in \mathbb{R},$$

and, moreover,

$$F(0) = G(0) = H(0) = 0.$$

Setting $y = 0$ in (1) we obtain

$$(2) \qquad |2F(\frac{x}{2}) - G(x)| \leq 2\delta, \qquad x \in \mathbb{R}.$$

For arbitrary $u \in \mathbb{R}$ and $v \leq 0$ the equation

$$z^2 - (u+v)z + v = 0$$

has two solutions x and y fulfilling the following equalities

$$u + v = x + y \qquad \text{and} \qquad v = xy.$$

Consequently,

$$(3) \qquad |2F(\frac{u+v}{2}) - G(u) - H(v)| \leq 2\delta, \qquad u \in \mathbb{R},\ v \leq 0.$$

Setting $u = 0$ in (3) we obtain

$$(4) \qquad |2F(\frac{v}{2}) - H(v)| \leq 2\delta, \qquad v \leq 0.$$

By virtue of (2), (3) and (4), for all $u \in \mathbb{R}$ and each $v \leq 0$, we have

$$|2F(\tfrac{u+v}{2}) - 2F(\tfrac{u}{2}) - 2F(\tfrac{v}{2})|$$
$$\leq |2F(\tfrac{u+v}{2}) - G(u) - H(v)| + |2F(\tfrac{u}{2}) - G(u)| + |2F(\tfrac{v}{2}) - H(v)| \leq 6\delta,$$

which can be rewritten to the following equivalent form

$$(5) \qquad |F(u+v) - F(u) - F(v)| \leq 3\delta, \qquad u \in \mathbb{R},\ v \leq 0.$$

According to the well-known theorem ([1], for example) there exists a uniquely determined additive function $A : \mathbb{R} \to \mathbb{R}$ such that

$$(6) \qquad |F(v) - A(v)| \leq 3\delta, \qquad v \leq 0.$$

Using also (2) we obtain

$$(7)\qquad |G(v)-A(v)| \le |G(v)-2F(\frac{v}{2})| + |2F(\frac{v}{2})-2A(\frac{v}{2})| \le 8\delta, \qquad v \le 0,$$

and, similarly, using (4) instead of (2)

$$(8)\qquad |H(v)-A(v)| \le 8\delta, \qquad v \le 0.$$

It follows from (3) (by putting $u=-v$) that

$$|G(-v)+H(v)| \le 2\delta, \qquad v \le 0.$$

For arbitrary $v>0$ we have

$$|G(v)-A(v)| \le |G(v)+H(-v)| + |A(-v)-H(-v)| \le 2\delta + 8\delta = 10\delta,$$

which together with (7) implies that

$$(9)\qquad |G(u)-A(u)| \le 10\delta, \qquad u \in \mathbb{R}.$$

According to (9) and (2)

$$|F(u)-A(u)| \le \frac{1}{2}|2F(u)-G(2u)| + \frac{1}{2}|G(2u)-A(2u)| \le 6\delta, \qquad u \in \mathbb{R}.$$

Putting $y=1$ and $x=v>0$ in (1) we get

$$|2F(\frac{v+1}{2}) - G(1) - H(v)| \le 2\delta,$$

and, consequently,

$$|H(v)-A(v)| \le |H(v)+G(1)-2F(\frac{v+1}{2})| + 2|F(\frac{v+1}{2})-A(\frac{v+1}{2})| + |G(1)-A(1)| \le 24\delta.$$

Therefore,

$$|F(x)-A(x)| \le 6\delta, \quad |G(x)-A(x)| \le 10\delta \quad \text{and} \quad |H(x)-A(x)| \le 24\delta, \qquad x \in \mathbb{R}.$$

Now, we are in a position to formulate our main result.

Theorem 0.1 *Let $\delta \ge 0$ be a fixed real number and let $f,g,h:\mathbb{R}\to\mathbb{R}$ be functions satisfying the following condition*

$$|2f(\frac{x+y}{2}) - g(x+y-xy) - h(xy)| \le \delta, \qquad x,y \in \mathbb{R}.$$

Then there exist functions $f_1, g_1, h_1 : \mathbb{R}\to\mathbb{R}$ fulfilling the equation

$$2f_1(\frac{x+y}{2}) - g_1(x+y-xy) - h_1(xy) = 0, \qquad x,y \in \mathbb{R}$$

and the following estimations

$$|f(x)-f_1(x)| \le 7\delta, \quad |g(x)-g_1(x)| \le 11\delta, \quad \text{and} \quad |h(x)-h_1(x)| \le 24\delta, \quad x \in \mathbb{R}.$$

Proof. Let F, G, H and A have the same meaning as above and let $d = 2f(0) - g(0) - h(0)$. As we observed $|d| \leq \delta$. We define functions f_1, g_1 and h_1 by the formulas

$$f_1(x) = A(x)+f(0)-d, \quad g_1(x) = A(x)+g(0)-d, \quad h_1(x) = A(x)+h(0),\ x \in \mathbb{R}.$$

Then

$$2f_1(\frac{x+y}{2}) - g_1(x+y-xy) - h_1(xy) = 2f(0) - g(0) - h(0) - d = 0,$$

and

$$|f(x) - f_1(x)| = |f(x) - f(0) + d - A(x)| \leq |F(x) - A(x)| + |d| \leq 7\delta, \quad x \in \mathbb{R},$$

$$|g(x) - g_1(x)| \leq |G(x) - A(x)| + |d| \leq 11\delta,$$

$$|h(x) - h_1(x)| = |H(x) - A(x)| \leq 24\delta,$$

as required. □

Remark 0.1 *The assertion of Theorem 1 says, in another words, that pexiderized Jensen-Hosszú equation of the form*

$$2f(\frac{x+y}{2}) = g(x+y-xy) + h(xy), \qquad x, y \in \mathbb{R},$$

is stable in the Hyers-Ulam sense.

If we take $\delta = 0$ we obtain the solution of the pexiderized Jensen-Hosszú equation.

Colorallary 0.1 *Functions $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy pexiderized Jensen-Hosszú equation if and only if there exist an additive function $A : \mathbb{R} \to \mathbb{R}$ and real constants a and b such that*

$$(10) \quad f(x) = A(x) + a, \quad g(x) = A(x) + b, \quad h(x) = A(x) + 2a - b, \qquad x \in \mathbb{R}.$$

Proof. Putting $\delta = 0$, from the proof of our Theorem, we get

$$f(x) = A(x) + f(0) - d, \quad g(x) = A(x) + g(0) - d, \quad h(x) = A(x) + h(0),\ x \in \mathbb{R},$$

where $d = 2f(0) - g(0) - h(0)$. If $a = f(0) - d,\ b = g(0) - d$ we obtain hence (10). On the other hand, functions defined by (10) fulfilled the equation

$$2f(\frac{x+y}{2}) = g(x+y-xy) + h(xy), \qquad x, y \in \mathbb{R}.$$

□

Remark 0.2 *Our main result generalizes an earlier author's result on the Hyers-Ulam stability of the Jensen-Hosszú equation [2].*

References

[1] Marek Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen Inequality*, PWN, Uniwersytet Slaski, Warszawa-Krakw-Katowice, 1985

[2] Z. Kominek, *On a Jensen-Hosszú equation, I*; Annales Math. Silesianae, 23(2009), 57-60.

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