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A Common Fixed Point Theorem Of Two Random Operators Using Random Ishikawa Iteration Scheme

R. A. Rashwan[1]

Abstract

In this paper, we discuss the convergence of random Ishikawa iteration scheme to a common random fixed point for a certain class of random operators in Banach spaces.

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1 Introduction and Preliminaries

In linear spaces there are two general iterations which have been successfully applied to fixed point problems of operators and also for obtaining solutions of operator equations. These are Ishikawa scheme [15] and Mann iteration scheme [18]. In [1], [12], [22], [23], and [25] it has shown that for a mapping T satisfying certain conditions, if the sequence of Mann iterates converges, then it converges to a fixed point of T. In [20], [21], the author has shown that for a pair of mappings S, T satisfying some contractive conditions, if the sequence of Mann and Ishikawa iterates associated with S or T is convergent, then its limit point is a common fixed point of S and T.

On the other hand random fixed points have attracted much attention, some of the recent literatures in random fixed points may be noted in [2, 3, 4, 5, 6, 7, 8, 9, 10, 16, 17, 22] and many others.

The study of random fixed point theory is playing an increasing role in Mathematics and applied Sciences. At present, it received considerable attention due to enormous applications in many important areas such as Nonlinear Analysis, Probability Theory and for the study of random equations arising in various applied areas.

[1]Assiut University, Faculty of Science, Department of Mathematics, Assiut, Egypt, e-mail: rr_ rashwan54@yahoo.com

The aim of this paper is to prove a random common fixed point which is stochastic generalization of the results in [20, 21]. Also we extend the results of [19], for a pair of mappings in a separable Banach space using Ishikawa iteration process.

We first review the following concepts, which are essential for our study in this paper.

Throughout this paper, (Ω, Σ) denotes a measurable space consisting of a set Ω and sigma algebra Σ of subsets of Ω, X stands for a separable Banach space and C is a nonempty subset of X.

Definition 1.1 *A mapping $f : \Omega \to C$ is said to be measurable if $f^{-1}(B \bigcap C) \in \sum$ for every Boral subset B of X.*

Definition 1.2 *A function $F : \Omega \times C \to C$ is said to be a random operator if $F(.,x) : \bigcap \longrightarrow C$ is measurable for every $x \in C$.*

Definition 1.3 *A measurable mapping $g : \Omega \to C$ is said to be random fixed point of the random operator $F : \Omega \times C \to C$, if $F(t, g(t)) = g(t)$ for all $t \in \Omega$.*

Definition 1.4 *A random operator $F : \Omega \times C \to C$ is said to be continuous if, for fixed $t \in \Omega$, $F(t,.) : C \to C$ is continuous.*

The Ishikawa iteration scheme was introduce in [15]. We define the randem Ishikawa iteration scheme in an analogous manner as follows: Let $S, T : \Omega \times C \to C$ be two operators on a nonempty convex subset C of a separable Banach space X. Then the sequence $\{x_n\}$ of random Ishikawa iterates associated with S or T is defined as follows:

$$\text{Let } x_0 : \Omega \to C \text{ by any given measurable mapping.} \tag{1}$$

$$\begin{aligned} x_{n+1}(t) &= (1-\alpha_n)x_n(t) + \alpha_n S(t, y_n), \\ y_n(t) &= (1-\beta_n)x_n(t) + \beta_n S(t, x_n), \ for \ \ n > 0, \ t \in \Omega \end{aligned} \tag{2}$$

or

$$\begin{aligned} x_{n+1}(t) &= (1-\alpha_n)x_n(t) + \alpha_n T(t, y_n), \\ y_n(t) &= (1-\beta_n)x_n(t) + \beta_n T(t, x_n), \ for \ \ n > 0, \ t \in \Omega \end{aligned} \tag{3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfying the following conditions

(i) $0 \le \alpha_n, \beta_n \le 1 \quad \forall \ n > 0$

(ii) $\lim_{n\to\infty} \beta_n = 0$

(iii) $\lim_{n\to\infty} \alpha_n = h, \quad 0 < h < 1.$

If $\beta_n = 0$, the Ishikawa iterates process reduces to the Mann iterates process.

2 Main Results

We present our main result in the form of the following theorem.

Theorem 2.1 *Let C be a nonempty, closed and convex subset of a separable Banach space X, and $S,T : \Omega \times C \to C$ be two continuous random operators defined on C satisfying one of the following: inequalities for all $x, y \in C$ and $t \in \Omega$:*

(I) $\|S(t,x) - T(t,y)\| \leq$
$$a\|x-y\| + b[\|x - S(t,x)\| + \|y - T(t,y)\|] + c[\|x - T(t,y)\| + \|y - S(t,x)\|],$$
$$1 - b - c > 0, \;\; a > 0, \;\; b \geq 0, \;\; c \geq 0.$$

(II) $\|S(t,x) - T(t,y)\| \leq$
$$q \max\{\|x-y\|, \|x - S(t,x)\| + \|y - T(t,y)\|, \|x - T(t,y)\| + \|y - S(t,x)\|\},$$
$$0 < q < 1.$$

(III) $\|S(t,x) - T(t,y)\| \leq$
$$\alpha \max\{\beta\|x-y\|, \|x - S(t,x)\|, \|y - T(t,y)\|, \|x - T(t,y)\|, \|y - S(t,x)\|\},$$
where $\alpha, \beta \geq 0, \;\; 0 \leq \alpha < 1.$

(IV) $\|S(t,x) - T(t,y)\| \leq$
$$q \max\{\|x-y\|, \|x - S(t,x)\|, \|y - T(t,y)\|, \|x - T(t,y)\|, \|y - S(t,x)\|\},$$
$$0 < q < 1.$$

If the sequence $\{x_n\}$ of random Ishikawa iterates associated with S or T (3), satisfying (i)-(iii) converges, then it converges to a common random fixed point of S and T. Moreover, if (IV) holds, then this common fixed point is unique.

Proof. We may assume that the sequence $\{x_n\}$ defined by (3) is a pointwise convergent, that is for all $t \in \Omega$

$$\lim_{n\to\infty} x_n(t) = u(t).$$

Since X is a separable Banach space, for any random operator $A : \Omega \to X$ and any measurable mapping $f : \Omega \to C$, the mapping $x(t) = A(t, f(t))$, is measurable mapping [14].

Since $x(t)$ is measurable and C is convex, it follows that $\{x_n\}$ constructed in the random iteration from (1), (3) is a sequence of measurable mappings. Hence $x : \Omega \to C$ being limit of measurable mapping sequence is also measurable.

Suppose first that $S(t, x(t)) = x(t)$ for $x(t) \in C$. Then putting $x(t) = y(t) = u(t)$ into any of the inequalities $(I) - (IV)$, we easily see that $S(t, u(t)) = u(t)$. Similarly $T(t, u(t)) = u(t)$ implies $S(t, u(t)) = u(t)$. Now, let $\{x_n\}$ be a sequence of Ishikawa iterates associated with S such that $\lim_{n\to\infty} x_n(t) = u(t)$.

From (3), $x_{n+1}(t) - x_n(t) = \alpha_n(S(t, y_n) - x_n(t))$. Since $\lim_{n\to\infty} x_n(t) = u(t)$ we have

$\|x_{n+1}(t)-x_n(t)\|\to 0$. The condition (III) implies $\|S(t,y_n)-x_n(t)\|\to 0$. It follows that $\|u(t)-S(t,y_n)\|\to 0$.

If S,T satisfy (I), then

$$\begin{aligned}\|S(t,y_n)-T(t,u(t))\| &\leq a\|y_n(t)-u(t)\|\\ &+ b[\|y_n(t)-S(t,y_n)\|+\|u(t)-T(t,u(t))\|]\\ &+ c[\|y_n(t)-T(t,u(t))\|+\|u(t)-S(t,y_n)\|].\end{aligned} \tag{4}$$

From (3) we have:

$$\|y_n(t)-u(t)\|\leq(1-\beta_n)\|x_n(t)-u(t)\|+\beta_n\|S(t,x_n)-u(t)\|, \tag{5}$$

$$\begin{aligned}\|y_n(t)-S(t,y_n)\|\leq{}&(1-\beta_n)\|x_n(t)-S(t,y_n)\|\\ &+\beta_n\|S(t,x_n)-S(t,y_n)\|,\end{aligned} \tag{6}$$

$$\begin{aligned}\|y_n(t)-T(t,u(t))\|\leq{}&(1-\beta_n)\|x_n(t)-T(t,u(t))\|\\ &+\beta_n\|S(t,x_n)-T(t,u(t))\|.\end{aligned} \tag{7}$$

substituting (5), (6) and (7) in (4), we get

$$\begin{aligned}\|S(t,y_n)-T(t,u(t))\| &\leq a[(1-\beta_n)\|x_n(t)-u(t)\|+\beta_n\|S(t,x_n)-u(t)\|]\\ &+ b[(1-\beta_n)\|x_n(t)-S(t,y_n)\|+\beta_n\|S(t,x_n)-S(t,y_n)\|\\ &+ \|u(t)-T(t,u(t))\|]\\ &+ c[(1-\beta_n)\|x_n(t)-T(t,u(t))\|\\ &+ \beta_n\|S(t,x_n)-T(t,u(t))\|+\|u(t)-S(t,y_n)\|].\end{aligned} \tag{8}$$

If S,T satisfy (II), then

$$\begin{aligned}\|S(t,y_n)-T(t,u(t))\| \leq{}& q\max\{\|y_n(t)-u(t)\|,\\ &\|y_n(t)-S(t,y_n)\|+\|u(t)-T(t,u(t))\|,\\ &\|y_n(t)-T(t,u)\|+\|u(t)-S(t,y_n)\|\}.\end{aligned} \tag{9}$$

substituting (5), (6) and (7) in (9) one gets

$$\begin{aligned}\|S(t,y_n)-T(t,u(t))\| &\leq q\max\{(1-\beta_n)\|x_n(t)-u(t))\|+\beta_n\|S(t,x_n)-u(t)\|,\\ &\quad (1-\beta_n)\|x_n(t)-S(t,y_n)\|+\beta_n\|S(t,x_n)-S(t,y_n)\|\\ &+ \|u(t)-T(t,u(t))\|,\\ &\quad (1-\beta_n)\|x_n(t)-T(t,u(t))\|+\beta_n\|S(t,x_n)-T(t,u(t))\|\\ &+ \|u(t)-S(t,y_n)\|\}.\end{aligned} \tag{10}$$

If S,T satisfy (III), then

$$\begin{aligned}\|S(t,y_n)-T(t,u(t))\| \leq{}& \alpha\max\{\beta\|y_n(t)-u(t))\|,\ \|y_n(t)-S(t,y_n))\|,\\ &\|u(t)-T(t,u(t))\|,\\ &\|y_n(t)-T(t,u(t))\|,\ \|u(t)-S(t,y_n)\|\}.\end{aligned} \tag{11}$$

substituting (5), (6) and (7) in (11) one gets:

$$\begin{aligned} \|S(t,y_n) - T(t,u(t))\| \quad \leq \quad & \alpha \max\{\beta(1-\beta_n)\|x_n(t) - u(t))\| + \beta_n\|S(t,x_n) - u(t)\|, \\ & (1-\beta_n)\|x_n(t) - S(t,y_n))\| + \beta_n\|S(t,x_n) - S(t,y_n)\|, \\ & \|u(t) - T(t,u(t))\|, \\ & (1-\beta_n)\|x_n(t) - T(t,u(t))\| + \beta_n\|S(t,x_n) - T(t,u(t))\|, \\ & \|u(t) - S(t,y_n)\|\}. \end{aligned} \tag{12}$$

If T, S satisfy (IV), then obviously satisfy (II) as well.

Taking in (8), (10), (12), the limit as $n \to \infty$ we have

$$\|u(t) - T(t,u(t))\| \quad \leq \quad \lambda\|u(t) - T(t,u(t))\|, \tag{13}$$

where $\lambda = \max\{b+c, q, \alpha\} < 1$. Then $T(t,u(t)) = u(t)$. Similarly, we can shaw that $S(t,u(t)) = u(t)$.

In order to show the uniqueness of $u(t)$ in the case (IV), suppose that $v(t), (v(t)) \neq u(t))$ is another common fixed point of S and T, then, using (IV), we have

$$\begin{aligned} \|u(t) - v(t)\| &= \|S(t,u(t)) - T(t,v(t))\| \\ &\leq q\max\{\|u(t) - Su(t)\|, \|u(t) - S(t,v(t)\|, \\ &\quad \|v(t) - Tv(t)\|, \|u(t) - Tv(t)\|, \|v(t) - Sv(t)\|\} \\ &\leq q\|u(t) - v(t)\|, \end{aligned}$$

where $u(t) = v(t)$ follows. □

Remark 2.1 *In the deterministic case*

1. *Our Theorem gives the results in [20], [21]*
2. *Theorem 1 in [19] is special case of our theorem with* $\beta_n = 0$*.*
3. *For* $S = T$ *with* $\alpha_n = t$, $0 < t < 1$, $\beta_n = 0$ *our result gives Theorem 3 of Ćirić [11].*
4. *It extends Theorem 1 of Rhoades [23].*

Remark 2.2 *The uniqueness of a common fixed point does not hold in the case of the conditions (I)-(III) as shown by the example of* $S = T = I_d$*, where* I_d *does denotes the identity mapping.*

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