Bulletin of International Mathematical Virtual Institute
ISSN 1840-4367
Vol. 1(2011), 21-26

Former
Bulletin of Society of Mathematicians Banja Luka
ISSN 0354-5792, ISSN 1986-521X (o)
Vol. 18 (2011), 25-30

DEDEKIND PARTIAL GROUPOIDS FOR ANTI-ORDERED SETS[1]

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Abstract

We associate with every anti-ordered set $((X,=,\neq),\alpha)$ with $\alpha\cap\alpha^{-1}=\emptyset$ a partial groupoid $((X,=,\neq),\cdot)$ in such a way that $(x,y)\in\alpha \Longleftrightarrow x\cdot y = y$ and $(x,y)\bowtie\alpha \Longleftrightarrow x\cdot y = x$ for two elements $x,y\in X$ such that $x\neq y$.

AMS Mathematics Subject Classification (2010): Primary: 03F65; Secondary: 06A06, 06A11

Key words and phrases: Constructive mathematics, set with apartness, anti-ordered set, Dedekind partial groupoid

1 Introduction

It is well known that in the Classical theory Dedekind's definition of lattices as algebras with two operations satisfying commutativity, associativity and absorption, is equivalent to the definition of lattices as partially ordered sets in which every two elements have greatest lower bound (g.l.b. or inf) and least upper bound (l.u.b. or sup). As a matter of fact, this equivalence holds at the level of semilattices (see e.g. [1], [6]). Recall that a meet semilattice is a poset $(S,\leq)$ such that every two elements have g.l.b. Then the operation

$$x\wedge y = inf\{x,y\}$$

is commutative, associative and idempotent; we say that $(S,\wedge)$ is a Dedekind semilattice. Conversely, if $(S,\wedge)$ is a commutative and idempotent semigroup, i.e., a Dedekind semilattice, then by defining

$$x\leq y \Longleftrightarrow x\wedge y = x,$$

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we obtain a meet semilattice $(S, \leq)$. Similarly, the correspondence

$$x \vee y = sup\{x, y\},$$

$$x \leq y \Longleftrightarrow x \vee y = y,$$

establishes a bijection between join semilattices and Dedekind semilattices. As a matter of fact the above bijections can be lifted to the categorial level: the categories of meet semilattices and join semilattices and Dedekind semilattices are isomorphic.

In the Classical theory of ordered sets connection between ordered sets and some kind of groupoids are interesting for researches. For example, Fiala and Novak in [5] describe a connection between so called o-groupid and partially ordered sets. Neggers in [8] introduced the internal operation between elements of partially ordered set and provided an axiomatic characterization of the groupoids in this way. Neggers and Kim in [9] define a partial order in an arbitrary semigroup and relate it to partially ordered groupoids.

Setting of this investigation is Bishop's constructive mathematics. In this note we suggest a Dedekind-like construction for arbitrary anti-ordered set instead of meet semilattices, by associating with every anti-ordered set a certain groupoid, which we call the Dedekind groupoid of the anti-ordered set. Our construction reduces to the conventional construction of the internal operation only in the case of chains.

2 Preliminaries

This investigation is in Bishop's constructive mathematics in sense of books [2], [3], [4], [7] and papers [10], [11], [12]. Let $(X, =, \neq)$ be a constructive set (i.e. it is a relational system with the relation "$\neq$ "). The diversity relation "$\neq$ " is a binary relation on X, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z.$$

If it satisfies the following condition

$$(\forall x, z \in X)(x \neq z \Longrightarrow (\forall y \in X)(x \neq y \vee y \neq z)),$$

it called *apartness* (A. Heyting).

For subset A of X we say that it is *strongly extensional subset* of X if and only if

$$x \in A \Longrightarrow (\forall y \in X)(x \neq y \vee y \in A).$$

Follows Bridges and Vita's definition for subsets A and B of X we say that set A is set-set apartness from B, and it is denoted by $A \bowtie B$, if and only if $(\forall x \in A)(\forall y \in B)(x \neq y)$. We set $x \bowtie B$ instead $x \bowtie Y$, and, of course, $x \neq y$ instead $\{x\} \bowtie \{y\}$. With $A^C = \{x \in X : x \bowtie A\}$ we denote apartness complement of A.

For a function $f : (X, =, \neq) \longrightarrow (Y, =, \neq)$ we say that it is a *strongly extensional* if and only if $(\forall a, b \in X)(f(a) \neq f(b) \Longrightarrow a \neq b)$.

A relation σ on set X is called a quasi-antiorder relation on X if

$$\sigma \subseteq \neq, \ \sigma \subseteq \sigma * \sigma.$$

If X is a semigroup, then σ has to compatible with the semigroup operation in the following way

$$(\forall a, b, x \in X)(((xa, xb) \in \sigma \vee (ax, bx) \in \sigma) \Longrightarrow (a, b) \in \sigma).$$

A quasi-antiorder relation α on set X is an *anti-order relation* on X if it satisfies yet another condition

$$\alpha \cup \alpha^{-1} = \neq .$$

If X is a semigroup, then α has to compatible with the semigroup operation.

For the necessary undefined notions, the reader is referred to books [2]-[4], [7] and to papers [10]-[12].

3 Dedekind partial groupoids for anti-ordered sets

This section we will begin with the following definition:

For given anti-ordered set $((X, =, \neq), \alpha)$, we define the relation $\cdot \subseteq (X \times X) \times X$ by the following way: If $(x, y) \in \alpha$, then $x \cdot y = y$ and if $(x, y) \bowtie \alpha$, then $x \cdot y = x$.

This definition is not a generalization of the operation $\wedge$ mentioned above. Let us note that for every $x \in X$ holds $x \cdot x = x$ because $(x, x) \bowtie \alpha$ holds for any $x \in X$.

In the following two lemmas we give important connection between the anti-order α and this relation on X.

Lemma 3.1 *Let $x, y \in X$ be any two elements $x, y \in X$ such that $x \neq y$, and $\alpha \cap \alpha^{-1} = \emptyset$. Then, the following conditions are equivalent:*

(1) $(x, y) \bowtie \alpha$;

(2) $x \cdot y = x$;

(3) $x \cdot y = y \cdot x = x$.

Proof: (1) $\Longrightarrow$ (3). Assume that $x \neq y$ and $(x, y) \bowtie \alpha$. Then, we have to have $(y, x) \in \alpha$ and, so on $y \cdot x = x$. Therefore, by definition we have $xy = x = yx$.

(3) $\Longrightarrow$ (2) Validity of this implication is obvious.

(2) $\Longrightarrow$ (1) Let (u, v) be an arbitrary pair of α. Then, we have $(u, x) \in \alpha$ or

$(x, y) \in \alpha$ or $(y, v) \in \alpha$. Since in the second case we have $x \cdot y = y \neq x$, which is impossible, we have $(x, y) \neq (u, v) \in \alpha$. This means $(x, y) \bowtie \alpha$. $\square$

Colorallary 3.1 *$x \cdot y = y \cdot x \Longleftrightarrow ((x, y) \bowtie \alpha \vee (y, x) \bowtie \alpha)$ for any two elements $x, y \in X$ such that $x \neq y$.*

Proof: The first part of assertion follows from Lemma 3.1. Further on, for elements $x, y \in X$ such that $x \cdot y = y \cdot x$ we have

$$\begin{aligned} x \neq y &\Longleftrightarrow (x, y) \in \alpha \ \vee \ (y, x) \in \alpha \\ &\Longrightarrow (x \cdot y = y = y \cdot x) \vee (y \cdot x = x = x \cdot y) \\ &\Longrightarrow (y, x) \bowtie \alpha \ \vee \ (x, y) \bowtie \alpha. \end{aligned}$$ $\square$

Colorallary 3.2 *The structure $(X, \cdot)$ is commutative if and only if (X, α^C) is a chain.*

Lemma 3.2 *Let $x, y \in X$ be any two elements $x, y \in X$ such that $x \neq y$, and $\alpha \cap \alpha^{-1} = \emptyset$. Then, the following conditions are equivalent:*
(4) $(x, y) \in \alpha$;
(5) $x \cdot y \neq x$;
(6) $x \cdot y = y \cdot x \wedge x \cdot y \neq x$.

Proof: (4) $\Longrightarrow$ (5). If $(x, y) \in \alpha$, then $x \cdot y = y \neq x$.

(5) $\Longrightarrow$ (4). Assume that $x \cdot y \neq x$. From $x \neq y$, follows $(x, y) \in \alpha$ or $(y, x) \in \alpha$. The second case is impossible because $(y, x) \in \alpha$ implies $(x, y) \bowtie \alpha$. Indeed, if (u, v) is an arbitrary element of α we have to

$$\begin{aligned} (u, v) \in \alpha &\Longrightarrow (u, x) \in \alpha \ \vee \ (x, y) \in \alpha \ \vee \ (y, v) \in \alpha \\ &\Longrightarrow (x, y) \neq (u, v) \in \alpha \text{ (because } \neg((x, y) \in \alpha) \text{ holds by hypothesis).} \end{aligned}$$

So, $(x, y) \bowtie \alpha$ implies $x \cdot y = x$ by definition. Least is in contradiction with $x \cdot y \neq x$. Therefore, we have $(x, y) \bowtie \alpha$.

(5) $\Longrightarrow$ (6) Assume $x \cdot y \neq x$. Then $(x, y) \in \alpha$ by (4) and $(y, x) \bowtie \alpha$ by hypothesis $\alpha \cap \alpha^{-1} = \emptyset$. So, from $x \cdot y = y$ and $y \cdot x = y$ we conclude $x \cdot y = y = y \cdot x$. Finally, we have $x \cdot y \neq x$ and $x \cdot y = y \cdot x$.

(6)$\Longrightarrow$ (5). Immediately follows. $\square$

Theorem 3.1 *(a) The relation '$\cdot$' is a partial function from $X \times X$ into X.*
(b) If $\alpha \cap \alpha^{-1} = \emptyset$, then the partial function '$\cdot$' is a strongly extensional function from $X \times X$ into X.
So, if $\alpha \cap \alpha^{-1} = \emptyset$, then the structure $((X, =, \neq), \cdot)$ is a idempotent partial groupoid.

Proof: (a). (i) Since $(x = x' \wedge y = y' \wedge (x, y) \in \alpha) \Longrightarrow (x', y') \in \alpha$, we have

$$x \cdot y = y = y' = x' \cdot y'.$$

(ii) Suppose that $x = x' \wedge y = y' \wedge (x, y) \bowtie \alpha$. Thus, we have $(x', y') \bowtie \alpha$ and therefore

$$x' \cdot y' = x' = x = x \cdot y.$$

So, the relation '$\cdot$' is a partially function from $X \times X$ into X.

(b). At the other hand, suppose that $x \cdot y' \neq x \cdot y$ for some elements $x, y, y' \in X$. Then

(i) $x \cdot y \neq x \cdot y' \wedge (x, y) \in \alpha \wedge (x, y') \in \alpha$
$\Longrightarrow x \cdot y \neq x \cdot y' \wedge x \cdot y = y \wedge x \cdot y' = y'$
$\Longrightarrow y \neq y'$;

(ii) $x \cdot y \neq x \cdot y' \wedge (x, y) \bowtie \alpha \wedge (x, y') \bowtie \alpha$
$\Longrightarrow x \cdot y \neq x \cdot y' \wedge x \cdot y = x \wedge x \cdot y' = x$
$\Longrightarrow x \neq x$ (It is impossible!);

(iii) $x \cdot y \neq x \cdot y' \wedge (x, y) \in \alpha \wedge (x, y') \bowtie \alpha$
$\Longrightarrow x \cdot y \neq x \cdot y' \wedge ((x, y') \in \alpha \vee (y', y) \in \alpha) \wedge (x, y') \bowtie \alpha$
$\Longrightarrow x \cdot y \neq x \cdot y' \wedge (y', y) \in \alpha$
$\Longrightarrow y \neq y'$;

(iv) $x \cdot y \neq x \cdot y' \wedge (x, y) \bowtie \alpha \wedge (x, y') \in \alpha$
$\Longrightarrow x \cdot y \neq x \cdot y' \wedge ((x, y) \in \alpha \vee (y, y') \in \alpha) \wedge (x, y) \bowtie \alpha$
$\Longrightarrow x \cdot y \neq x \cdot y' \wedge (y, y') \in \alpha$
$\Longrightarrow y \neq y'$.

(v) $x \cdot y \neq x \cdot y' \wedge (y', x) \in \alpha \wedge (y, x) \in \alpha$
$\Longrightarrow x \cdot y \neq x \cdot y' \wedge y \cdot x = x \wedge y' \cdot x = x$
$\Longrightarrow x \neq x$ (Is is impossible!);

(vi) $x \cdot y \neq x \cdot y' \wedge (y', x) \bowtie \alpha \wedge (y, x) \in \alpha$
$\Longrightarrow x \cdot y \neq x \cdot y' \wedge (y', x) \bowtie \alpha \wedge (y, y') \in \alpha \vee (y', x) \in \alpha)$
$\Longrightarrow y \neq y'$;

(vii) $x \cdot y \neq x \cdot y' \wedge (y, x) \bowtie \alpha \wedge (y', x) \in \alpha$
$\Longrightarrow x \cdot y \neq x \cdot y' \wedge (y, x) \bowtie \alpha \wedge ((y', y) \in \alpha \vee (y, x) \in \alpha)$
$\Longrightarrow y' \neq y$;

(viii) $x \cdot y \neq x \cdot y' \wedge (y, x) \bowtie \alpha \wedge (y', x) \bowtie \alpha$
$\Longrightarrow x \cdot y \neq x \cdot y' \wedge x \cdot y = y \cdot x = y \wedge y' \cdot x = x \cdot y' = y'$
$\Longrightarrow y \neq y'$;

(ix) $x \cdot y \neq x \cdot y' \wedge (y, x) \bowtie \alpha \wedge (x, y') \bowtie \alpha$
$\Longrightarrow (x \cdot y \neq x \vee x \neq x \cdot y') \wedge (y, x) \bowtie \alpha \wedge (x, y') \bowtie \alpha$
$\Longrightarrow ((x, y) \in \alpha \vee (x, y') \in \alpha) \wedge (y, x) \bowtie \alpha \wedge (x, y') \bowtie \alpha$
$\Longrightarrow (x, y) \in \alpha \wedge (y, x) \bowtie \alpha \wedge (x, y') \bowtie \alpha$
$\Longrightarrow ((x, y') \in \alpha \vee (y', y) \in \alpha) \wedge (y, x) \bowtie \alpha \wedge (x, y') \bowtie \alpha$
$\Longrightarrow (y, y') \in \alpha$
$\Longrightarrow y \neq y'$;

(x) $x \cdot y \neq x \cdot y' \wedge (y, x) \bowtie \alpha \wedge (x, y') \in \alpha$
$\Longrightarrow x \cdot y \neq x \cdot y' \wedge x \cdot y = y \cdot x = y \wedge x \cdot y' = y'$
$\Longrightarrow y \neq y'$;

(xi) $x \cdot y \neq x \cdot y' \wedge (y, x) \in \alpha \wedge (x, y') \in \alpha$

$\Longrightarrow x \cdot y \neq x \cdot y' \wedge ((y, y') \in \alpha \vee (y', x) \in \alpha) \wedge (x, y') \in \alpha$
$\Longrightarrow x \cdot y \neq x \cdot y' \wedge (y, y') \in$
$\Longrightarrow y \neq y'$.

We prove the implication $y \cdot x \neq y' \cdot x \Longrightarrow y \neq y'$ on similar way. So, the partial function '$\cdot$' is a strongly extensional function. Finally, the structure $(X, =, \neq)$ is an idempotent partial groupoid under this internal operation. $\square$

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Received by the editors March 06, 2010