SOME NOTES ON QUASI-ANTIORDERS
AND COEQUALITY RELATIONS

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Abstract

It is known that each quasi-antiorder on anti-ordered set $X$ induces coequality $q$ on $X$ such that $X/q$ is an anti-ordered set. The converse of this statement also holds: Each coequality $q$ on a set $X$ such that $X/q$ is anti-ordered set induces a quasi-antiorder on $X$. In this paper we give proofs that the families of all coequality relations $q$ on $X$ and the family of all quasi-antiorder relation on set $X$ are completely lattices.

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1 Introduction and preliminary

This short investigation, in Bishop’s constructive mathematics in the sense of well-known books [2], [4], [6] and [18] and Bogdanić, Romano and Vinčić’s paper [3], Jojić and Romano’s paper [6], and Romano’s papaers [7]-[16], is continuation of forthcoming the second author’s papers [17]. Bishop’s constructive mathematics is developed on Constructive Logic - logic without the Law of Excluded Middle $P ∨ ¬P$. Let us note that in the Constructive Logic the ‘Double Negation Law’ $¬¬P$ does not hold, but the following implication $P → ¬¬P$ does even in the Minimal Logic. Since the Constructive Logic is a part of the Classical Logic, these results, in the Constructive mathematics, are compatible with suitable results in the Classical mathematics. Let us recall that the following deduction principle $A ∨ B, ¬B ⊢ A$ is acceptable in the Constructive Logic.

Let $(X, =, ≠)$ be a set, where the relation $≠$ is a binary relation on $X$, called diversity on $X$, which satisfies the following properties:

$$¬(x ≠ x), \ x ≠ y ⇒ y ≠ x, \ x ≠ y \land y = z ⇒ x ≠ z.$$
Following Heyting, if the following implication
\[ x \neq z \implies x \neq y \lor y \neq z \]
holds, the diversity \( \neq \) is called \textit{apartness}. Let \( x \) be an element of \( X \) and \( A \) a subset of \( X \). We write \( x \succeq A \) if and only if \( (\forall a \in A)(x \neq a) \), and \( A^C = \{ x \in X : x \succeq A \} \).

In \( X \times X \) the equality and diversity are defined by \( (x,y) = (u,v) \iff x = u \land y = v \), \( (x,y) \neq (u,v) \iff x \neq u \lor y \neq v \), and equality and diversity relations in power-set \( \wp(X \times X) \) of \( X \times X \) by

\[
\alpha =_2 \beta \iff (\forall (x,y) \in X \times X)((x,y) \in \alpha \iff (x,y) \in \beta),
\]

\[
\alpha \neq _2 \beta \iff \exists (x,y) \in X((x,y) \in \alpha \land (x,y) \not\in \beta) \lor \exists (x,y) \in X((x,y) \in \beta \land (x,y) \not\in \alpha).
\]

Let us note that the diversity relation \( \neq _2 \) is not an apartness relation in general case.

\textbf{Example I:} (1) The relation \( \neg (=) \) is an apartness on the set \( \mathbb{Z} \) of integers.

(2) The relation \( q \), defined on the set \( \mathbb{Q} \) by

\[
(f,g) \in q \iff (\exists k \in \mathbb{N})(\exists n \in \mathbb{N})(m \geq n \implies |f(m) - g(m)| > k^{-1}),
\]

is an apartness relation. ♦

A relation \( q \) on \( X \) is a coequality relation ([7]-[9]) on \( X \) if and only if it is consistent, symmetric and cotransitive:

\[
q \subseteq \neq, \quad q = q^{-1}, \quad q \subseteq q \star q,
\]

where "" \( \star \) "" is the operation of relations \( \alpha \subseteq X \times X \) and \( \beta \subseteq X \times X \), called filled product ([8], [9], [12]-[15]) of relations \( \alpha \) and \( \beta \), are relation on \( X \) defined by

\[
(a,c) \in \beta \star \alpha \iff (\forall b \in X)((a,b) \in \alpha \lor (b,c) \in \beta).
\]

For further study of coequality relation we suggest to read articles [8], [11], [13]-[16] (Specially, in articles [10], [12], [13] and [14], the author researches coequality relations compatible with the algebraic operations.) In article [7] and [8], problems of existence of compatible equality and coequality relations on set with apartness are discussed. In article [9], the author has proved the following: If \( e \) is an equivalence on set \( X \), then there exists the maximal coequality relation \( q \) on \( X \) compatible with \( e \) in the following sense:

\[
equiv {e} \circ q \subseteq q \quad \text{and} \quad q \circ e \subseteq q.
\]

Opposite to the previous, if \( q \) is a coequality relation on set \( X \), then the relation \( q^C = \{(x,y) \in X \times X : (x,y) \not\in q \} \) is an equivalence on \( X \) compatible with \( q \) ([8], [11]), and we can ([11]) construct the factor-set \( X/(q^C) = \{aqC : a \in X \} \) with:

\[
aqC =_1 bqC \iff (a,b) \succeq q, \quad aqC \neq _1 bqC \iff (a,b) \in q.
\]
Also, we can ([8],[11]) construct the factor-set \( X/q = \{ aq : a \in X \} \): If \( q \) is a coequality relation on a set \( X \), then \( X/q \) is a set with:

\[
aq = bq \iff (a,b) \trianglerighteq q, \ aq \neq bq \iff (a,b) \in q.
\]

It is easily to check that \( X/q \cong X/(qC,q) \). besides, it is clear that the mapping \( \pi : X \to X/q \), defined by \( \pi(x) = xq \), is a strongly extensional surjective function.

Subset \( C(x) = \{ y \in X : y \neq x \} \) satisfies the following implication:

\[
y \in C(x) \land z \in X \implies y \neq z \lor z \in C(x).
\]

It is called a principal strongly extensional subset of \( X \) such that \( x \asymp C(x) \). Following this special case, for a subset \( A \) of \( X \), we say that it is a strongly extensional subset of \( X \) if and only if the following implication

\[
x \in A \land y \in X \implies x \neq y \lor y \in A
\]

holds.

**Examples II:** (1) ([7]) Let \( T \) be a set and \( J \) be a subfamily of \( \wp(T) \) such that

\[
\emptyset \in J, \ A \subseteq B \land B \in J \implies A \in J, \ A \cap B \in J \implies A \in J \lor B \in J.
\]

If \( (X_i)_{i \in T} \) is a family of sets, then the relation \( q \) on \( \prod_{i \in T} X_i(\neq \emptyset) \), defined by \((f,g) \in q \iff \{ s \in T : (f(s) = g(s)) \in J \) is a coequality relation on the Cartesian product \( \prod_i X_i \).

(2) A ring \( R \) is a local ring if for each \( r \in R \), either \( r \) or \( 1-r \) is a unit, and let \( M \) be a module over \( R \). The relation \( q \) on \( M \), defined by \((x,y) \in q \) if there exists a homomorphism \( f : M \to R \) such that \( f(x-y) \) is a unit, is a coequality relation on \( M \).

(3) ([11]) Let \( T \) be a strongly extensional consistent subset of semigroup \( S \), i.e. let \((\forall x,y \in S)(xy \in T \implies x \in T \land y \in T) \) holds. Then, relation \( q \) on semigroup \( S \), defined by \((a,b) \in q \) if and only if \( a \neq b \land (a \in T \lor b \in T) \), is a coequality relation on \( S \) compatible with semigroup operation in the following sense:

\[
(\forall x,y,a,b \in S)((axy, xby) \in q \implies (a,b) \in q).
\]

(4) Let \((R, =, \neq, +, 0, -, 1)\) be a commutative ring. A subset \( Q \) of \( R \) is a coideal of \( R \) if and only if

\[
0 \asymp Q, \ -x \in Q \implies x \in Q, \ x + y \in Q \implies x \in Q \lor y \in Q,
\]

\[
xy \in Q \implies x \in Q \land y \in Q.
\]

Coideals of commutative ring with apartness were first defined and studied by Ruitenburg 1982 in his dissertation. After that, coideals (anti-ideals) are studied by A.S. Troelstra and D. van Dalen in their monograph [18]. This author proved, in 1988, if \( Q \) is a coideal of a ring \( R \), then the relation \( q \) on \( R \), defined
by \((x, y) \in q \iff x - y \in Q\), satisfies the following properties:

(a) \(q\) is a coequality relation on \(R\);

(b) \((\forall x, y, u, v \in R)((x + u, y + v) \in q \implies (x, y) \in q \lor (u, v) \in q)\);

(c) \((\forall x, y, u, v \in R)((xu, yv) \in q \implies (x, y) \in q \lor (u, v) \in q)\).

A relation \(q\) on \(R\), which satisfies the property (a)-(c), is called anticongruence on \(R\) or coequality relation compatible with ring operations. If \(q\) is an anti-congruence on a ring \(R\), then the set \(Q = \{x \in R : (x, 0) \in q\}\) is a coideal of \(R\).

As in [12],[13], [14] and [15] a relation \(\alpha\) on \(X\) is antiorder on \(X\) if and only if

\[
\alpha \subseteq \neq, \alpha \subseteq \alpha \ast \alpha, \neq \subseteq \alpha \cup \alpha^{-1} \text{ (linearity)}.
\]

Let \(g\) be a strongly extensional mapping of anti-ordered set from \((X, =, \neq, \alpha)\) into \((Y, =, \neq, \beta)\). For \(g\) we say that it is:

(i) isotone if \((\forall a, b \in X)((a, b) \in \alpha = \implies (g(a), g(b)) \in \beta)\) holds;

(ii) reverse isotone if \((\forall a, b \in X)((g(a), g(b)) \in \beta = \implies (a, b) \in \alpha)\) holds.

A relation \(\sigma\) on \(X\) is a quasi-antiorder ([11]–[16]) on \(X\) if

\[
\sigma \subseteq (\alpha \subseteq) \neq, \sigma \subseteq \sigma \ast \sigma.
\]

It is clear that each coequality relation \(q\) on set \(X\) is a quasi-antiorder relation on \(X\), and the apartness is a trivial anti-order relation on \(X\). It is easy to check that if \(\sigma\) is a quasi-antiorder on \(X\), then ([10]) the relation \(q = \sigma \cup \sigma^{-1}\) is a coequality relation on \(X\). The notion of quasi-antiorder is defined for first time in article [8], and the notion of anti-order relation is defined for the first time in article [10]. Those relations and their properties are investigated by Baroni in [1], Bogdanić, Jojić and Romano in [3], Jojić and Romano in [6], and van Plato in [19] also.

**Examples III:** Let \(a\) and \(b\) be elements of semigroup \((S, =, \neq, \cdot)\). Then ([11]), the set \(C_{(a)} = \{x \in S : x \approx SaS\}\) is a consistent subset of \(S\) such that :

- \(a \approx C_{(a)}\);
- \(C_{(a)} \neq \emptyset \implies 1 \in C_{(a)}\);
- Let \(a\) be an invertible element of \(S\). Then \(C_{(a)} = \emptyset\);
- \((\forall x, y \in S)(C_{(a)} \subseteq C_{(xy)})\);
- \(C_{(a)} \cup C_{(b)} \subseteq C_{(ab)}\).

Let \(a\) be an arbitrary element of a semigroup \(S\) with apartness. The consistent subset \(C_{(a)}\) is called a principal consistent subset of \(S\) generated by \(a\). We introduce relation \(f\), defined by \((a, b) \in f \iff b \in C_{(a)}\). The relation \(f\) has the
following properties ([11, Theorem 7]):
- $f$ is a consistent relation ;
- $(a, b) \in f \implies (\forall x, y \in S)((x ay, b) \in f)$;
- $(a, b) \in f \implies (\forall n \in N)((a^n, b) \in f)$;
- $(\forall x, y \in S)((a, xby) \in f \implies (a, b) \in f)$ ;
- $(\forall x, y \in S)((a, xay) \in f)$.

We can construct the cotransitive relation $c(f) = \bigcap^n f$ as cotransitive fulfillment of the relation $f$ ([8]-[11],[15]). As consequences of these assertions we have the following results. The relation $c(f)$ satisfies the following properties:
- $c(f)$ is a quasi-antiorder on $S$ ;
- $(\forall x, y \in S)((a, xay) \bowtie c(f))$;
- $(\forall n \in N)((a, a^n) \bowtie c(f))$ ;
- $(\forall x, y \in S)((a, b) \in c(f) \implies (xay, b) \in c(f))$ ;
- $(\forall n \in N)((a, b) \in c(f) \implies (an, b) \in c(f))$ ;
- $(\forall x, y \in S)((a, xby) \in c(f) \implies (a, b) \in c(f))$. ♦

For a given anti-ordered set $(X, =, \neq, \alpha)$ is essential to know if there exists a coequality relation $q$ on $X$ such that $X/q$ is an anti-ordered set. This plays an important role for studying the structure of anti-ordered sets. The following question is natural: If $(X, =, \neq, \alpha)$ is an anti-ordered set and $q$ a coequality relation on $X$, is the factor-set $X/q$ anti-ordered set? Naturally, anti-order on $X/q$ should be the relation $\Theta$ on $X/q$ defined by means of the anti-order $\alpha$ on $X$ such that $\Theta = \{(xq, yq) \in X/q : (x, y) \in \alpha\}$, but it is not held in general case. The following question appears: Is there coequality relation $q$ on $X$ for which $X/q$ is an anti-ordered set such that the natural mapping $\pi : X \to X/q$ is reverse isotope? The concept of quasi-antiorder relation was introduced by this author in his papers [8] and [9]-[16] (Particularly, in articles [10] and [14], the author investigated anti-ordered algebraic systems with apartness.). According to Lemma 0 in [12], if $(X, =, \neq)$ is a set and $\sigma$ is a quasi-antiorder on $X$, then ([12, Lemma 1]) the relation $q$ on $X$, defined by $q = \sigma \cup \sigma^{-1}$ , is a coequality relation on $X$, and the set $X/q$ is an anti-ordered set under anti-order $\Theta$ defined by $(xq, yq) \in \Theta \iff (x, y) \in \sigma$. So, according to results in [12] and [13], each quasi-antiorder $\sigma$ on an ordered set $X$ under anti-order $\alpha$ induces an coequality relation $q =_2 \sigma \cup \sigma^{-1}$ on $X$ such that $X/q$ is an anti-ordered set under $\Theta$ . (For a further study of quasi-antiorders on anti-ordered set we refer to papers [12], [13] and forthcoming the author’s paper [17].) In paper [14] we proved that the converse of this statement also holds. If $(X, =, \neq, \alpha)$ is an anti-ordered set and $q$ coequality relation on $X$, and if there exists an order relation $\Theta_1$ on $X/q$ such that the $(X/q, =_1, \neq_1, \Theta_1)$ is an anti-ordered and the mapping $\pi : X \to X/q$ is reverse isotope (so-called regular coequality), then there exists a quasi-antiorder $\sigma$ on $X$ such that $q =_2 \sigma \cup \sigma^{-1}$. So, each regular coequality $q$ on a set $(X, =, \neq, \alpha)$
induces a quasi-antiorder on $X$. Besides, connections between the family of all quasi-antiorders on $X$, the family of coequality relations on $X$, and the family of all regular coequality relations $q$ on $X$ are given.

**Lemma 1.1** Let $\tau$ be a quasi-antiorder on set $X$. Then $x \tau (\tau x)$ is a strongly extensional subset of $X$, such that $x \nabla x \tau (x \nabla \tau x)$, for each $x \in X$. Besides, the following implication $(x, z) \in \tau \implies x \tau \cup \tau z = X$ holds for each $x, z$ of $X$.

**Proof:** From $\tau \subseteq \neq$ it follows $x \bowtie x \tau$. Let $yx \in \tau$ holds, and let $z$ be an arbitrary element of $X$. Thus, $(x, y) \in \tau$ and $(x, z) \in \tau \cup (z, y) \in \tau$. So, we have $z \in x \tau \cup y \neq z$. Therefore, $x \tau$ is a strongly extensional subset of $X$ such that $x \bowtie x \tau$.

The proof that $\tau x$ is a strongly extensional subset of $X$ such that $x \bowtie \tau x$ is analogous. Besides, the following implication $(x, z) \in \tau \implies x \tau \cup \tau z = X$ holds for each $x, y$ of $X$. Indeed, if $(x, z) \in \tau$ and $y$ is an arbitrary element of $X$, then $(\forall y \in X)((x, y) \in \tau \cup (y, z) \in \tau)$. Thus, $X = x \tau \cup \tau z$.

Let $\tau$ be a quasi-antiorder on set $X$. Then for every pair $(x, z)$ of $\tau$ there exists a pair $(A_x, B_z)$ of strongly extensional subsets of $X$ such that $x \bowtie A_x \land z \bowtie B_z$ and $X = A_x \cup B_z$ and $x \in B_z \land z \in A_x$.

**Example IV:** If $A$ is a strongly extensional subset of $X$, then the relation $\sigma$ on $X$, defined by $(x, y) \in \sigma \iff x \in A \land x \neq y$, is a quasi-antiorder relation on $X$.

**Proof:** It is clear that $\sigma$ is a consistent relation on $X$. Assume $(x, z) \in \sigma$ and let $y$ be an arbitrary element of $X$. Thus, $x \neq y \lor y \neq z$. If $x \neq y$ and $x \in A$, then $(x, y) \in \sigma$. If $y \neq z$ and $x \in A$, by strongly extensionality of $A$, we have $y \neq z$ and $x \in A$ and $x \neq y \lor y \in A$. In the case of $y \neq z \land x \in A \land x \neq y$ we have again $(x, y) \in \sigma$; in the case of $y \neq z$ and $x \in A$ and $y \in A$ we have $(y, z) \in \sigma$. So, the relation $\sigma$ is a cotransitive relation. Therefore, relation $\sigma$ is a quasi-antiorder relation on $X$. Further on, we have:

\[
x \in A \implies x \sigma = C(x), \quad \neg (x \in A) \implies x \sigma = \emptyset;
\]

\[
y \in A \implies \sigma y = C(y) \cap A, \quad y \bowtie A \implies \sigma y = A. \quad \blackdiamond
\]

**2 Main Results**

In the following proposition we give a connection between the family $\Im(X)$ of all quasi-antiorders on set $X$ and the family $q(X)$ of all coequality relation on $X$: Footnotes should be avoided. Drawings should be prepared as

For a set $(X, =, \neq, \alpha)$ by $\Re(X, \alpha)$ we denote the family of all regular coequality relations $q$ on $X$ with respect to $\alpha$, and by $\Im(X, \alpha)$ denotes the family of all quasi-antiorder relation on $X$ included in $\alpha$. In the following assertion we give another main result of this paper: Postscript files. Here is an example:
Let us note that families $\Im(X)$, $\Im(X,\alpha)$ and $q(X)$ are completely lattices. Indeed, in the following two theorems we give proofs for those facts:

**Theorem 2.1** If $\{\tau_k\}_{k \in J}$ is a family of quasi-antiorders on a set $(X,=,\neq)$, then $\bigcup_{k \in J} \tau_k$ and $c(\bigcap_{k \in J} \tau_k)$ are quasi-antiorders in $X$. So, the families $\Im(X)$ and $\Im(X,\alpha)$ are completely lattices.

**Proof:** (1) Let $\{\tau_k\}_{k \in J}$ is a family of quasi-antiorders on a set $(X,=,\neq)$ and let $x,z$ be an arbitrary elements of $X$ such that $(x,z) \in \bigcup_{k \in J} \tau_k$. Then, there exists $k$ in $J$ such that $(x,z) \in \tau_k$. Hence, for every $y \in X$ we have $(x,y) \in \tau_k \lor (y,z) \in \tau_k$. So, $(x,y) \in \bigcup_{k \in J} \tau_k \lor (y,z) \in \bigcup_{k \in J} \tau_k$. On the other hand, for every $k$ in $J$ holds $\tau_k \subseteq \neq$. From this we have $\bigcup_{k \in J} \tau_k \subseteq \neq$. So, we can put $\sup\{\tau_k : k \in J\} = \bigcup_{k \in J} \tau_k$.

(2) Let $R(\subseteq \neq)$ be a relation on a set $(X,=,\neq)$. Then for an inhabited family of quasi-antiorders under $R$ there exists the biggest quasi-antiorder relation under $R$. That relation is exactly the relation $c(R)$. In fact:

By (1), there exists the biggest quasi-antiorder relation on $X$ under $R$. Let $Q_R$ be the inhabited family of all quasi-antiorder relation on $X$ under $R$. With $(R)$ we denote the biggest quasi-antiorder relation $\bigcup Q_R$ on $X$ under $R$. On the other hand, the fulfillment $c(R) = \bigcap_{n \in N} ^n R$ of the relation $R$ is a cotransitive relation on set $X$ under $R$. Therefore, $c(R) \subseteq (R)$ holds. We need to show that $(R) \subseteq c(R)$. Let $\tau(\subseteq (R) = \bigcup Q_R)$ be a quasi-antiorder relation in $X$ under $R$. Firstly, we have $\tau \subseteq R = ^1 R$. Assume $(x,z) \in \tau$. Then, out of $\forall y \in X)((x,y) \in \tau \lor (y,z) \in \tau)$ we conclude that for every $y$ in $X$ holds $(x,y) \in R \lor (y,z) \in R$, i.e. holds $(x,z) \in R \lor R = ^2 R$. So, we have $\tau \subseteq ^2 R$. Now, we will suppose that $\tau \subseteq ^n R$, and suppose that $(x,z) \in \tau$. Then, $\forall y \in X)((x,y) \in \tau \lor (y,z) \in \tau)$ implies that $(x,y) \in R \lor (y,z) \in ^n R$ holds for every $y \in X$. Therefore, $(x,z) \in ^{n+1} R$. So, we have $\tau \subseteq ^{n+1} R$. Thus, by induction, we have $\tau \subseteq \bigcap ^n R$. Let us remember that $\tau$ is an arbitrary quasi-antiorder on $X$ under $R$. Hence, we proved that $(R) = \bigcup Q_R \subseteq c(R)$. If $\{\tau_k\}_{k \in J}$ is a family of quasi-antiorders on a set $(X,=,\neq)$, then $c(\bigcap_{k \in J} \tau_k)$ is a quasi-antiorder in $X$, and we can set $\inf\{\tau_k : k \in J\} = c(\bigcap_{k \in J} \tau_k)$.

**Theorem 2.2** Let $(X,=,\neq)$ be a set with apartness. The family $q(X)$ is a completely lattice.

**Proof:** If $\{q_k : k \in \Lambda\}$ is a family of coequality relations on $X$, then $\bigcup q_k$ and $c(\bigcap q_k)$ are coequality relations on $X$ such that $\forall k \in \Lambda)q_k \subseteq \bigcup q_k)$ and $\forall k \in \Lambda)c(\bigcap q_k \subseteq q_k)$. Since $\bigcup q_k$ is the minimal extension of every $q_k$ we can put $\sup\{q_k : k \in \Lambda\} = \bigcup q_k$, and since $c(\bigcap q_k)$ is the maximal coequality relation under $\bigcap q_k \subseteq q_k$ we can set $\inf\{q_k : k \in \Lambda\} = c(\bigcap q_k)$.
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Some Notes on Quasi-antiorders and Coequality relations


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