AN ISOMORPHISM THEOREM FOR ANTI-ORDERED SETS

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Abstract

In this paper we show some kind of isomorphism theorem for ordered sets under antiorders. Let \((X, =_X, \neq_X, \alpha)\) and \((Y, =_Y, \neq_Y, \beta)\) be ordered sets under antiorders, where the apartness \(\neq_Y\) is tight. If \(\varphi : X \rightarrow Y\) is reverse isotone strongly extensional mapping, then there exists a strongly extensional and embedding reverse isotone bijection

\[ ((X, =_X, \neq_X, \alpha, c(R))/q, =_1, \neq_1, \gamma) \longrightarrow (\text{Im}(\varphi), =_Y, \neq_Y, \beta) \]

where \(c(R)\) is the biggest quasi-antiorder relation on \(X\) under \(R = \alpha \cap \text{Coker}(\varphi)\), \(q = c(R) \cup c(R)^{-1}\) and \(\gamma\) is an antiorder induced by the quasi-antiorder \(c(R)\). If the condition \(\alpha \cap \alpha^{-1} = \emptyset\) holds, then the above bijection is isomorphism.

1 Introduction

1.1 Setting. The arguments in this paper conform to Constructive mathematics in the sense of Bishop ([2]). So, our setting is Bishop’s constructive mathematics, mathematics developed with Constructive logic (or Intuitionistic logic ([24])) - logic without the Law of Excluded Middle \(P \lor \neg P\). We have to note that 'the crazy axiom' \(\neg P \Rightarrow (P \Rightarrow Q)\) is included in the Constructive logic. Precisely, in Constructive logic the 'Double Negation Law' \(P \iff \neg\neg P\) does not hold, but the following implication \(P \Rightarrow \neg\neg P\) holds even in Minimal logic. In Constructive logic 'Weak Law of Excluded Middle' \(\neg P \lor \neg\neg P\) does not hold also. It is interesting, in Constructive logic the following deduction principle \(A \lor B, \neg A \vdash B\) holds, but this is impossible to prove without 'the crazy axiom'. As Intuitionistic logic is a fragment of Classical logic, our arguments should be valid from a classical point of view.

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1.2 Set with apartness. Let \((X, =, \neq)\) be a set, where "\(\neq\)" is a apartness \([2],[4],[8],[12],[14],[23],[24]\). Apartness is a binary relation on \(X\) which satisfies the following properties:

\[\neg(x \neq x), \quad x \neq y \implies y \neq x, \quad x \neq z \implies (\forall y \in X)(x \neq y \vee y \neq z)\]

for every \(x, y\) and \(z\) in \(X\). The apartness is compatible with the equality in the following sense \((\forall x, y, z \in X)(x = y \land y \neq z \implies x \neq z)\). The apartness \(\neq\) is tight \([23],[24]\) if and only if \(\neg(x \neq y) \implies x = y\) for any element \(x\) and \(y\) in \(X\). Let \(x\) be an element of \(X\) and \(A\) subset of \(X\). We write \(x \triangleright\!
lessgtr A\) if \((\forall a \in A)(x \neq a)\), and \(A^C = \{x \in X : x \triangleright\!
lessgtr A\}\).

A relation \(q \subseteq X \times X\) is coequality relation on \(X\) \([10],[12]\) if

\[q \subseteq \neq, \quad q^{-1} = q, \quad (\forall x, z \in X)((x, z) \in q \implies (\forall y \in X)((x, y) \in q \lor (y, z) \in q)).\]

The relation \(q^C = \{(x, y) \in X \times X : (x, y) \triangleright\!
lessgtr q\}\) is an equality on \(X\) compatible with \(q\), in the following sense

\[(\forall a, b, c \in X)((a, b) \in q^C \land (b, c) \in q \implies (a, c) \in q)\]

\([18]\), Theorem 1\). We can \([10],[12]\) construct factor-sets

\[(X/(q^C, q), =_1, \neq_1) = \{aq^C : a \in X\} \quad \text{and} \quad (X/q, =_1, \neq_1) = \{aq : a \in X\},\]

where

\[aq^C =_1 bq^C \iff (a, b) \triangleright\!
lessgtr q, \quad aq^C \neq_1 bq^C \iff (a, b) \in q, \quad aq =_1 bq \iff (a, b) \in q, \quad aq \neq_1 bq \iff (a, b) \in q.
\]

It is easy to check that \(X/(q^C, q) \cong X/q\).

**Examples I:** (1) The relation \(\neg(=)\) is an apartness on the set \(\mathbb{Z}\) of integers.

(2) \([8]\) The relation \(q\), defined on the set \(Q^N\) by

\[(f, g) \in q \iff (\exists k \in N)(\exists n \in N)(m \geq n \implies |f(m) - g(m)| > k^{-1}),\]

is a coequality relation.

(3) \([8]\) A ring \(R\) is a local ring if for each \(r \in R\), either \(r\) or \(1 - r\) is a unit. Let \(M\) be a module over \(R\). Then the relation \(q\) on \(M\), defined by \((x, y) \in q\) if there exists a homomorphism \(f : M \longrightarrow R\) such that \(f(x - y)\) is a unit, is a coequality relation on \(M\).

(4) \([12]\) Let \(T\) be a set and \(J\) be a subfamily of \(\wp(T)\) such that \(\emptyset \notin J\), \(A \subseteq B \land B \in J \implies A \in J\), \(A \cap B \in J \implies A \in J \lor B \in J\). If \((X_t)_{t \in T}\) is a family of sets, then the relation \(q\) on \(\prod X_t\) \((\neq \emptyset)\), defined by \((f, g) \in q \iff \{s \in T : f(s) = g(s)\} \in J\), is a coequality relation on the Cartesian product \(\prod X_t\).
1.3 Algebraic structures with apartness. For a function

\[ f : (X, =, \neq) \longrightarrow (Y, =, \neq) \]

we say that it is:
(a) \([\text{8}]\) strongly extensional if and only if \((\forall a, b \in X)(f(a) \neq f(b) \implies a \neq b)\);
(b) \([\text{7}]\) an embedding if and only if \((\forall a, b \in X)(a \neq b \implies f(a) \neq f(b))\).

In general, all functions in this text are strongly extensional functions. For example, if \(\omega : X \times X \longrightarrow X\) is an internal binary operation on \(X\), then must be:

\[(\forall a, b, x, y \in X)(\omega(a, b) \neq \omega(x, y) \implies (a, b) \neq (x, y)).\]

**Examples II**: Let \((S, =, \neq, \cdot)\) be a semigroup with apartness. Let us note that the internal operation "\(\cdot\)" is a strongly extensional function in the following sense:

\[(\forall x, a, b \in S)((ax \neq bx \implies a \neq b) \land (xa \neq xb \implies a \neq b)).\]

A subset \(T\) of semigroup \(S\) is a right consistent subset of \(S\) \([\text{3}]\) of \(S\) if and only if

\[(\forall x, y \in S)(xy \in T \implies y \in T);\]

a subset \(T\) of \(S\) is a left consistent subset of \(S\) \([\text{3}]\) of \(S\) if and only if

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a subset \(T\) of \(S\) is a consistent subset of \(S\) \([\text{3}]\) of \(S\) if and only if

\[(\forall x, y \in S)(xy \in T \implies x \in T \land y \in T).\]

Let \(q\) be a coequality relation on a semigroup \(S\) such that

\[(\forall a, b, y \in S)((ay, by) \in q \implies (a, b) \in q).\]

Then we say that it is a left anticongruence on \(S\). If for \(q\) holds

\[(\forall a, b, x \in S)((xa, xb) \in q \implies (a, b) \in q)\]

then \(q\) is a right anticongruence on \(S\). The coequality relation \(q\) on \(S\) is an anticongruence on \(S\), or relation compatible with semigroup operation on \(S\), if and only if it is a left and right anticongruence.

1. Let \(e\) and \(f\) be idempotents of a semigroup \(S\) with apartness. Then:
(a) the set \(X(e) = \{a \in S : ae \neq a\}\) is a strongly extensional right consistent subset of \(S\);
(b) the set \(Y(e) = \{b \in S : eb \neq b\}\) is a strongly extensional left consistent subset of \(S\);
(c) the set \(P(e) = \{a \in S : e \bowtie aS\}\) is a strongly extensional left ideal of \(S\);
(d) the set \(Q(e) = \{a \in S : e \bowtie a\bowtie S\}\) is a strongly extensional right ideal of \(S\);
(e) the set \(R(e) = \{a \in S : e \bowtie a\bowtie S\}\) is a strongly extensional ideal of \(S\) such
that $e \bowtie R(e)$.

(f) the set $M(e) = X(e) \cup Y(e) \cup P(e) \cup R(e)$ is a strongly extensional completely prime subset of $S$ such that $e \bowtie M(e)$. Besides, if $e \neq f$, then $M(e) \cup M(f) = S$.

(2) Let $S = \{0\} \times [0,1] \subset R \times R$, where $R$ is the set of reals. The multiplication in $S$ is the coordinatewise usual multiplication. Then $S$ is a semigroup with apartness. The set $\{0\} \times [0,1]$ is an ideal of $S$ and the set $\{0\} \times [1/2,1]$ is a consistent subset of $S$.

(3) The set $S = \{(x,y) \in R \times R : x \geq 0 \land y \geq 0\}$ with the multiplication on $S$ defined by $(x,y)(a,b) = (xa, xb + y)$ is a semigroup with apartness. The subset $Q = \{(x,y) \in S : x > 0\}$ is a consistent subset of $S$ and filter in $S$.

(4) Let $T$ be a strongly extensional consistent subset of semigroup $S$, i.e. let $(\forall x,y \in S)(xy \in T \implies x \in T \land y \in T)$. Then relation $q$ on semigroup $S$, defined by $(a,b) \in q$ if and only if $a \neq b \land (a \in T \lor b \in T)$, is a coequality relation on $S$.

(5) ([18], Theorem 5) Let $q$ be a coequality relation on a semigroup $S$ with apartness. Then the relation $q^+ = \{(x,y) \in S \times S : (\exists a,b \in S^1)((axb, ayb) \in q)\}$ is an anticongruence on $S$ such that $q \subseteq q^+$. If $\rho$ is an anticongruence on $S$ such that $q \subseteq \rho$, then $q^+ \subseteq \rho$.

**Examples III:** Let $(R,=,\neq,+,-,1)$ be a commutative ring. A subset $Q$ of $R$ is a coideal of $R$ if and only if

$$0 \bowtie Q,$$

$$-x \in Q \implies x \in Q,$$

$$x + y \in Q \implies x \in Q \lor y \in Q,$$

$$xy \in Q \implies x \in Q \land y \in Q.$$  

Coideals of commutative ring with apartness where studied by Ruitenburg 1982 ([23]). After that, coideals (anti-ideals) studied by A.S. Troelstra and D. van Dalen in their monograph [24]. The author proved, in his paper [9], if $Q$ is a coideal of a ring $R$, then the relation $q$ on $R$, defined by $(x,y) \in q \iff x - y \in Q$, satisfies the following properties:

(a) $q$ is a coequality relation on $R$;

(b) $(\forall x,y,u,v \in R)((x + u, y + v) \in q \implies (x,y) \in q \lor (u,v) \in q)$;

(c) $(\forall x,y,u,v \in R)((xu, yv) \in q \implies (x,y) \in q \lor (u,v) \in q)$.

A relation $q$ on $R$, which satisfies the properties (a)-(c), is called anticongruence on $R$ ([9]) or relation compatible with ring operations. If $q$ is an anticongruence on a ring $R$, then the set $Q = \{x \in R : (x,0) \in q\}$ is a coideal of $R$ ([9]). Let $J$ be an ideal of $R$ and if $Q$ is a coideal of $R$. Wim Ruitenburg, in his dissertation ([23], page 33) first stated a demanded that $J \subseteq \neg Q$. This condition is equivalent with the following condition

$$(\forall x,y \in R)(x \in J \land y \in Q \implies x + y \in Q).$$

In this case we say that they are compatible ([11]) and we can construct the quotient-ring $R/JQ$. W.Ruitenburg, in his dissertation, first stated the question on the existence an ideal $J$ of $R$ compatible with a given coideal $Q$ and the
question on the existence of a coideal $Q$ of $R$ compatible with a given ideal $J$. If $e$ is a congruence on $R$, determined by the ideal $J$ and if $q$ is an anticonegruence on $R$, determined by $Q$, then $J$ and $Q$ are compatible if and only if

$$(\forall x, y, z \in R)(x, y) \in e \land (y, z) \in q \implies (x, z) \in q).$$

In this case we say that $e$ and $q$ are compatible.

(1) Let $R = (R, =, \neq, +, 0, \cdot, 1)$ be a commutative ring with apartness. Then the sets $\emptyset$ and $R = \{a \in R : a \neq 0\}$ are coideals of $R$. Let $a$ be an element of the ring $R$. Then the ideal $Ann(a)$ and the coideal $Can(a) = \{x \in R : ax \neq 0\}$ are compatible.

(2) Let $m$ and $i \in \{1, 2, ..., m-1\}$ be integers. We set $m\mathbb{Z}+i = \{mz+i : z \in \mathbb{Z}\}$. Then the set $\cup\{m\mathbb{Z}+i : i \in \{1, ..., m-1\}\}$ is a coideal of the ring $\mathbb{Z}$.

(3) Let $K$ be a Richman field and $x$ be an unknown variable under $K$. Then the set $C = \{f \in K[x] : f(0) \neq 0\}$ is a coideal of the ring $K[x]$.

(4) Let $R$ be a commutative ring. Then the set $B = R^N$ is a ring. For $n \in N$, the set $M = \{f \in B : f(n) \neq 0\}$ is a coideal of $B$.

(5) Let $R$ be a local ring. Then the set $M = \{a \in R : (\exists x \in R)(ax = 1)\}$ is a coideal of $R$.

(6) Let $S$ be a coideal of a ring $R$ and let $X$ be a subset of $R$. Then the set $[S : X] = \{a \in R : (\exists x \in X)(ax \in S)\}$ is a coideal of $R$.

(7) Let $H$ be a nonempty family of inhabited subsets of $T$. Then the set $S(H) = \{r \in \prod_{t \in T}R_t : (\exists A \in H)(A \cap Z(r) \neq \emptyset)\}$, where $Z(r) = \{t \in T : r(t) \neq 0\}$, is a coideal of the ring $\prod_{t \in T}R_t (\neq \emptyset)$.

1.3 Filed product. Let $\alpha \subseteq X \times Y$ and $\beta \subseteq Y \times Z$ be relations. As in [15], we define

$$\beta \ast \alpha = \{(x, z) \in X \times Z : (\forall y \in Y)((x, y) \in \alpha \land (y, z) \in \beta)\}.$$

For a relation $R \subseteq X \times X$ we put $^1R = R$, $^nR = R \ast R \ast ... \ast R$ ($n \geq 2$) and $c(R) = \bigcap_{n \in N}^nR$. In [14] and [15] this author proved that the relation $c(R)$ is a cotransitive relation under $R$. This relation is called the cotransitive fulfillment of $R$.

1.4 Goal of this paper. We will briefly recall the constructive definition of linear order and we will use a generalization of J. von Plato ([9]) and M.A. Baroni’s ([1]) excess relation for the definition of a partially ordered set. Let $X$ be a nonempty set. A binary relation $<$ (less than) on $X$ is called a linear order if the following axioms are satisfied for all elements $x$ and $y$:

$$\neg(x < y \land y < x),$$

$$x < y \implies (\forall z \in S)(x < z \lor z < y).$$

An example is the standard strict order relation $<$ on $R$, as described in [2], [4], [8] and [9]. For an axiomatic definition of the real number line as a constructive
ordered field, the reader is referred to [2], [4], [9]. A detailed investigation of linear orders in lattices can be found in [9]. The binary relation \( \leq \) on \( X \) is called an excess relation if it satisfies the following axioms:

\[
\neg(x \leq x),
\]
\[
x \not\leq y \implies (\forall z \in S)(x \not\leq z \lor z \not\leq y).
\]

Clearly, each linear order is an excess relation. As shown in [9], we obtain an apartness relation \( \neq \) and a partial order \( \leq \) on \( X \) by the following definitions:

\[
x \neq y \iff (x \not\leq y \lor y \not\leq x),
\]
\[
x \leq y \iff \neg(x \neq y).
\]

Note that the statement \( \neg(x \leq y) \implies x \not\leq y \) does not hold in general.

Let \( (X, =, \neq) \) be a set with apartness. A relation \( \alpha \subseteq X \times X \) is an antiorder relation on \( X \) if and only if

\[
\alpha \subseteq \neq,
\]
\[
(\forall x, y, z \in X)((x, z) \in \alpha \implies (x, y) \in \alpha \lor (y, z) \in \alpha),
\]
\[
(\forall x, y \in X)(x \neq y \implies (x, y) \in \alpha \lor (y, x) \in \alpha).
\]

A ordered set under an antiorder \( \alpha \) is a structure \( (X, =, \neq, \alpha) \) where \( \alpha \) is an antiorder relation on \( X \). Antiorder relation on a set was first defined by author in paper [14] and [16].

**Example IV**: Let \( S = \{a, b, c, d, e\} \) with multiplication defined by schema

\[
\begin{array}{c|ccccc}
\text{a} & b & c & d & e \\
\hline
a & a & a & c & d & e \\
b & a & b & c & d & e \\
c & b & a & c & d & e \\
d & c & a & b & c & d \\
e & e & c & d & e \\
\end{array}
\]

The relation \( \alpha \subseteq S \times S \), defined by \( \alpha = \{(a, b), (a, c), (a, e), (b, a), (b, c), (b, d), (b, e), (c, a), (c, b), (c, d), (d, a), (d, b), (d, c), (d, e), (e, a), (e, b), (e, c), (e, d)\} \) is an antiorder relation on semigroup \( S \). ♦

A relation \( \sigma \) on \( (X, =, \neq) \) is a quasi-antiorder relation on \( X \) if and only if

\[
\sigma \subseteq \neq, \\
(\forall x, y, z \in X)((x, z) \in \sigma \implies (x, y) \in \sigma \lor (y, z) \in \sigma).
\]

If there exists an antiorder \( \alpha \) on the set \( (X, =, \neq) \), different from \( \neq \), then we have to put a stronger demand in the definition of quasi-antiorder: \( \sigma \subseteq \alpha \) instead of \( \sigma \subseteq \neq \). A quasi-antiordered set is a structure \( (X, =, \neq, \sigma) \) where \( \sigma \) is a quasi-antiorder relation on \( X \). Note that if \( \sigma \) is a quasi-antiorder on \( X \), then \( \sigma^{-1} \) is a quasi-antiorder in \( X \) too. Indeed:
An isomorphism theorem for anti-ordered sets

(a) \( \sigma \subseteq \neq \implies \sigma^{-1} \subseteq \neq^{-1} \neq \) (because the relation \( \neq \) is symmetric);
(b) \((x, z) \in \sigma^{-1} \iff (z, x) \in \sigma\)
\[\implies (\forall y \in X)((z, y) \in \sigma \lor (y, x) \in \sigma)\]
\[\implies (\forall y \in X)((y, z) \in \sigma^{-1} \lor (x, y) \in \sigma^{-1})\]
\[\iff (\forall y \in X)((x, y) \in \sigma^{-1} \lor (y, z) \in \sigma^{-1}).\]

There is a theory of quasi-order relation in ordered semigroup. See, for example, papers [5] and [6]. In this paper we continue the research parallel relations of antiorder and quasi-antiorder.

**Example V**: Let \( S = \{a, b, c, d, e\} \) with multiplication defined by schema
\[
\begin{array}{|c|c|c|c|c|}
\hline
 & a & b & c & d & e \\
\hline
a & b & b & d & d & d \\
b & b & b & d & d & d \\
c & d & d & c & d & c \\
d & d & d & d & d & d \\
e & d & d & c & d & c \\
\hline
\end{array}
\]

Relation \( \alpha \), defined by \( \alpha = \{(a, c), (a, d), (a, e), (b, a), (b, c), (b, d), (b, e), (c, a), (c, b), (c, d), (c, e), (d, a), (d, e), (e, a), (e, b), (e, d)\} \), is an antiorder relation on semigroup \( S \). The relation \( \sigma = \{(a, c), (b, e), (c, a), (c, b), (c, d), (c, e), (d, e), (e, a), (e, b), (e, d)\} \) is a quasi-antiorder relation on semigroup \( S \). ♦

The notion of quasi-antiorder relation in set with apartness was introduced for first time in paper [14]. After that, quasi-antiorders are studied by this author in his paper [18], [19], [20], [21], [22]. Sometime, in the definition of antiorder relation on a set \((X, =, \neq)\), we add the condition \( \alpha \cap \alpha^{-1} = \emptyset \). In that case, in the definition of quasi-antiorder relation on the ordered set \((X, =, \neq, \alpha)\) under the antiorder \( \alpha \), we must add the following condition \( \sigma \cap \sigma^{-1} = \emptyset \). What is different between anti-order relation and excess relation? Clearly, an anti-order relation on set with tight apartness is an excess relation, and, opposite, an excess relation is an anti-order relation.

In this note we proved some kind of isomorphism theorem for ordered sets under antiorders. Let \((X, =_X, \neq_X, \alpha)\) and \((Y, =_Y, \neq_Y, \beta)\) be ordered sets under antiorders, where the apartness \( \neq_Y \) is tight. If \( \varphi : X \rightarrow Y \) is reverse isotone function, then there exists a strongly extensional, injective and embedding reverse isotone bijection
\[
((X, =_X, \neq_X, \alpha, c(R))/{q, =_1, \neq_1, \gamma}) \longrightarrow (Im(\varphi), =_Y, \neq_Y, \beta),
\]
where \( c(R) \) is the biggest quasi-antiorder relation on \( X \) under \( R = \alpha \cap \text{Coker}(\varphi) \), \( q = c(R) \cup c(R)^{-1} \) and \( \gamma \) is the antiorder induced by the quasi-antiorder \( c(R) \). If the condition \( \alpha \cap \alpha^{-1} = \emptyset \) holds, then the above bijection is an isomorphism.
1.5 References. For undefined notions and notations for Constructive mathematics we refer to the books [2], [4], [8], [23] and [24], and to papers [7], [11]-[20]. For classical and constructive semigroup theory we refer to [3], [5], [6] and [7], [17]-[20].

2 Preliminaries

In this section we start with the following explanations:

Remarks A.

(0) A relation $q$ on a set $(X, =, \neq)$ is a coequality relation on $X$ if and only if

$$q \subseteq \neq, \quad q^{-1} = q, \quad q \subseteq q \ast q.$$ 

(1) A relation $\alpha$ is an antiorder relation on a set $(X, =, \neq)$ if and only if

$$\alpha \subseteq \neq, \quad \alpha \subseteq \alpha \ast \alpha, \quad \neq \subseteq \alpha \cup \alpha^{-1}.$$ 

(2) A relation $\sigma$ on a ordered set $(X, =, \neq, \alpha)$ under an antiorder $\alpha$ is a quasi-antiorder relation on $X$ iff

$$\sigma \subseteq \alpha, \quad \sigma \subseteq \sigma \ast \sigma.$$ 

(3) Sometimes, in the definition of antiorder relation on set $(X, =, \neq)$, we add another condition

$$(\forall x, y \in X)((x, y) \in \alpha \implies \neg((y, x) \in \alpha)),$$

which is equivalent with the condition

$$\alpha \cap \alpha^{-1} = \emptyset.$$ 

In that case, in the definition of quasi-antiorder relation on the ordered set $(X, =, \neq, \alpha)$ under the antiorder $\alpha$, we must add the following condition

$$(\forall x, y \in X)((x, y) \in \sigma \implies \neg((y, x) \in \sigma)),$$

i.e. the demand

$$\sigma \cap \sigma^{-1} = \emptyset.$$ 

Let $(X, =, \neq, \alpha), (Y, =, \neq, \beta)$ be ordered sets under antiorders $\alpha$ and $\beta$ respective, $f : X \to Y$ a mapping from $X$ into $Y$. $f$ is called isotone if

$$(\forall x, y \in S)((x, y) \in \alpha \implies (f(x), f(y)) \in \beta).$$

$f$ is called reverse isotone if and only if

$$(\forall x, y \in S)((f(x), f(y)) \in \beta \implies (x, y) \in \alpha).$$
The mapping $f$ is called an isomorphism if it is injective and embedding, onto, isotone and reverse isotone. $X$ and $Y$ are called isomorphic, symbolically $X \cong Y$, if exists an isomorphism between them.

Remarks B.

B.1. Every isotope mapping $f : X \to Y$ satisfies the following condition:
(1) Let $x, y \in X$ and $x \neq_X y$. Then $(x, y) \in \sigma$ or $(y, x) \in \sigma$ by linearity of $\sigma$ and we have $(f(x), f(y)) \in \beta \subseteq \neq_Y$ or $(f(y), f(x)) \in \beta \subseteq \neq_Y$. So, the mapping $f$ is an embedding.
(2) Let $x, y \in X$ and $f(x) = f(y)$. Then $(f(x) \neq_Y f(y))$, and from this we conclude $¬((f(x), f(y)) \in \beta)$ and $¬((f(y), f(x)) \in \beta)$. Hence $¬((x, y) \in \alpha)$ and $¬((y, x) \in \alpha)$. Therefore $¬(x \neq_X y)$. If the apartness $\neq_X$ on set $X$ is tight, then $x = y$. So, in that case when the apartness is tight, the mapping $f$ is an injective.

B.2. Every reverse isotope mapping $f : X \to Y$ satisfies the following condition:
(3) Let $x, y \in X$ such that $f(x) \neq_Y f(y)$. Then $(f(x), f(y)) \in \beta$ or $(f(y), f(x)) \in \beta$ by linearity of $\beta$ and we have $(x, y) \notin \neq_X$ or $(y, x) \notin \neq_X$. So, the mapping $f$ is strongly extensional.
(4) Let $x =_X y$. Then $¬(x \neq_X y)$, i.e. then $¬((x, y) \in \alpha \cup \alpha^{-1})$. Suppose that $f(x) \neq_Y f(x)$, i.e. suppose that $(f(x), f(y)) \in \beta$ and $(f(y), f(x)) \in \beta$. Thus we conclude $(x, y) \in \alpha \cup \alpha^{-1}$ which is impossible. So, our proposition $f(x) \neq_Y f(x)$ is wrong, i.e. must holds $¬((f(x) \neq_Y f(x))$. If the apartness $\neq_Y$ is tight, then holds $f(x) =_Y f(y)$. So, in this case when the apartness $\neq_X$ is tight, antiorders are compatible with the function $f$.

**Lemma 0:** Let $\sigma$ be a quasi-antiorder relation on an anti-ordered set $(X, =_X, \neq_X, \alpha)$. Then $q = \sigma \cup \sigma^{-1}$ is a coequality relation on $X$ such that $(X/q, 1, \neq_1)$ is an ordered set under the antiorder relation $\beta$ defined by $(xq, yq) \in \beta \iff (x, y) \in \sigma$.

**Proof:** Let $(uq, vq)$ be an arbitrary element of $\beta$, i.e. let $(u, v) \in \sigma$. Since $\sigma \subseteq q$, we have $uq \neq_1 vq$. Therefore, $\beta \subseteq \neq_1$ (in $X/q$). Let $(xq, zq)$ and $yqX/q$, i.e. let $(x, z)$ and $yX$. Since $(x, y)/(y, z)$, we have $(xq, yq)$ or $(yq, zq)$. Let $(xq, yq) \in \beta$ and $aq, bq \in X/q$, i.e. $(x, y) \in \sigma$ and $a, b \in X$. Let $xq \neq_1 yq$, i.e. let $(x, y) \in q = \sigma \cup \sigma^{-1}$. Since $(x, y) \in \sigma$ or $(y, x) \in \sigma$, we have $(xq, yq) \in \beta$ or $(yq, xq) \in \beta$. So, the relation $\beta$ is linear. Therefore, the relation $\beta$ is an antiorder relation on $X/q$.

Now, suppose that $\sigma \cap \sigma^{-1} = \emptyset$. Then also $\beta \cap \beta^{-1} = \emptyset$. Indeed, let $(xq, yq) \in \beta$, i.e. let $(x, y) \in \sigma$. Then $¬((x, y) \in \sigma)$, i.e. then $¬((yq, xq) \in \beta)$.

**Example VI:** Let $S$, $\alpha$ and $\sigma$ as in the example II. Then the relation $q = \sigma \cup \sigma^{-1} = \{(a, e), (b, e), (c, a), (c, b), (c, d), (c, e), (e, a), (e, b), (e, d), (e, a), (e, b), (a, c), (b, c), (d, e), (c, e), (a, e), (b, e), (d, e)\}$ is an anticongruence on $S$. Then $aq = \{c, e\}$, $bq = \{c, e\}$, $cq = \{a, b, d, e\}$, $dq = \{c, e\}$, $eq = \{a, b, c, d\}$ and $S/q = \{\{c, e\}$, $\{a, b, d, e\}$, $\{a, b, c, d\}\}}}$. So, the relation $\beta$ is defined in the following way: $\beta = \{(c, e), (a, e), (b, e), (d, e)\}$.
\{(aq, eq), (bq, eq), (cq, aq), (cq, bq), (cq, dq), (eq, aq), (eq, bq), (eq, dq)\}.

\begin{corollary} \label{corollary:isotone_surjective}
The mapping \( \pi : X \rightarrow X/q \) is a reverse isotone surjective function.
\end{corollary}

\begin{lemma} \label{lemma:union_quasi_antiorder}
If \( \{\sigma_k\}_{k \in J} \) is a family of quasi-antiorders on a set \( (X, \equiv, \neq) \) relatively to a certain antiorder \( \alpha \), then \( \bigcup_{k \in J} \sigma_k \) is a quasi-antiorder in \( X \).
\end{lemma}

\begin{proof}
Let \((x, z)\) be an arbitrary elements of \( X \times X \) such that \((x, y) \in \bigcup_{k \in J} \sigma_k \). Then there exists \( k \) in \( J \) such that \((x, z) \in \sigma_k \). Hence for every \( y \in X \) we have \((x, y) \in \sigma_k \lor (y, z) \in \sigma_k \). So, \((x, y) \in \bigcup_{k \in J} \sigma_k \lor (y, z) \in \bigcup_{k \in J} \sigma_k \). At the other side, for every \( k \) in \( J \) holds \( \sigma_k \subseteq \alpha \). From this we conclude \( \bigcup_{k \in J} \sigma_k \subseteq \alpha \). \( \Box \)
\end{proof}

\section{The main results}

First, we show a construction of maximal quasi-antiorder under a given relation:

\begin{theorem} \label{theorem:maximal_quasi_antiorder}
Let \( R (\subseteq \neq) \) be a relation on a set \( (X, \equiv, \neq) \). Then there exists an inhabited family of all quasi-antiorders under \( R \) there exists the biggest quasi-antiorder relation \( \alpha \) on \( X \). That relation is exactly the relation \( c(R) \).
\end{theorem}

\begin{proof}
By Lemma \ref{lemma:union_quasi_antiorder}, there exists the biggest quasi-antiorder relation on \( X \) under \( R \). Let \( A_R \) be the inhabited family of all quasi-antiorder relation on \( X \) under \( R \). With \( (R) \) we denote the biggest quasi-antiorder relation \( \bigcup A_R \) on \( X \) under \( R \). The fulfillment \( c(R) = \bigcap_{n \in \mathbb{N}} \, ^n R \) of the relation \( R \) is a cotransitive relation on set \( X \) under \( R \). Therefore, \( c(R) \subseteq (R) \) holds.

We need to show that \( (R) \subseteq c(R) \). Let \( s \) be a quasi-antiorder relation in \( X \) under \( R \). First, we have \( s \subseteq R = 1 R \). Let \((x, z) \in s \). Then from \( (\forall y \in X)((x, y) \in s \lor (y, z) \in s) \) we conclude that for every \( y \in X \) holds \((x, y) \in R \lor (y, z) \in R \), i.e. holds \((x, z) \in R * R = 2 R \). So, \( s \subseteq 2 R \). Now, we will suppose that \( s \subseteq ^n R \) and let \((x, z) \in s \). Then from \( (\forall y \in X)((x, y) \in s \lor (y, z) \in s) \) implies that \((x, y) \in R \lor (y, z) \in R \) holds for every \( y \in X \). Therefore, \((x, z) \in ^{n+1} R \). So, we have \( s \subseteq ^{n+1} R \). Thus, by induction, we have \( s \subseteq \cap R \). Remember that \( s \) is an arbitrary quasi-antiorder on \( X \) under \( R \). Hence, we proved that \( (R) = \bigcup A_R \subseteq c(R) \). \( \Box \)
\end{proof}

\begin{corollary} \label{corollary:consistent_subset}
Let \((X, \equiv, \neq, \alpha)\) be an ordered set under an antiorder \( \alpha \). Then the family \( \mathcal{A} = \{\tau : \tau \, \text{is a quasi-antiorder on} \, X \, \text{under} \, \alpha\} \) is a complete lattice.
\end{corollary}

\begin{example}[\cite{18}]
Let \( a \) and \( b \) be elements of semigroup \( S \). Then \((18), \text{Theorem 6})\) the set \( C_{(a)} = \{x \in S : x \triangleleft \alpha S \alpha S\} \) is a consistent subset of \( S \) such that:
- \( a \triangleleft C_{(a)} \);
- \( C_{(a)} \neq \emptyset \implies 1 \in C_{(a)} \);
\end{example}
- Let $a$ be an invertible element of $S$. Then $C(a) = \emptyset$.
- $(\forall x, y \in S)(C(a) \subseteq C(xay))$;
- $C(a) \cup C(b) \subseteq C(ab)$.

Let $a$ be an arbitrary element of a semigroup $S$ with apartness. The consistent subset $C(a)$ is called a principal consistent subset of $S$ generated by $a$. We introduce relation $f$, defined by $(a, b) \in f \iff b \in C(a)$ and in the next assertion we will give some description of the relation $f$: The relation $f$ has the following properties ([17], Theorem 7)
- $f$ is a consistent relation;
- $(a, b) \in f \implies (\forall x, y \in S)((xay, b) \in f)$;
- $(a, b) \in f \implies (\forall n \in N)((a^n, b) \in f)$;
- $(\forall x, y \in S)((a, xby) \in f \implies (a, b) \in f)$;
- $(\forall x, y \in S)^{-1}((a, xay) \in f)$.

We can construct the cotransitive relation $c(f) = \bigcap_{n \in N} f^n$ as cotransitive fulfillment of the relation $f$ ([7],[14],[18]). As corollary of these assertions we have the following results: The relation $c(f)$ satisfies the following properties:
- $c(f)$ is a consistent relation on $S$;
- $c(f)$ is a cotransitive relation;
- $(\forall x, y \in S)((a, xay) \triangleright c(f))$;
- $(\forall n \in N)((a, an) \triangleright c(f))$;
- $(\forall x, y \in S)((a, b) \in c(f) \implies (xay, b) \in c(f))$;
- $(\forall n \in N)((a, b) \in c(f) \implies (a^n, b) \in c(f))$;
- $(\forall x, y \in S)((a, xby) \in c(f) \implies (a, b) \in c(f))$.

For an element $a$ of a semigroup $S$ and for $n \in N$ we introduce the following notations

$$A_n(a) = \{ x \in S : (a, x) \in f^n \}, \ A(a) = \{ x \in S : (a, x) \in c(f) \}$$

By the following results we will present some basic characteristics of these sets.
Let $a$ and $b$ be elements of a semigroup $S$. Then:
- $A_1(a) = C(a)$;
- $A_{n+1}(a) \subseteq A_n(a)$;
- $A_{n+1}(a) = \{ x \in S : S = A_n(a) \cup B_1(x) \}$ where $B_1(x) = \{ u \in S : (u, x) \in f \}$;
- $A(a) = \bigcap_{n \in N} A_n(a)$;
- $a \triangleright A(a)$;
- $A(a) \cup A(b) \subseteq A(ab)$;
- The set $A(a)$ is the maximal strongly extensional consistent subset of $S$ such that $a \triangleright A(a)$.

**Example VIII:** Let $S = \{ a, b, c, d \}$ be a ordered semigroup with the Cayley table and antiorder shown below:
\[ \alpha = \{(a, b), (a, c), (a, d), (b, a), (c, a), (c, b), (d, a), (d, b), (d, c)\} \]. For relation \( \alpha \) holds \( \alpha \cap \alpha^{-1} \neq \emptyset \). If we put \( \beta = \{(a, b), (a, c), (a, d), (c, b), (d, a), (d, c)\} \), then \( \beta \subset \alpha \) and \( \beta \cap \beta^{-1} = \emptyset \).

Let \( f : X \rightarrow Y \) be a strongly extensional function. It is easy to verify that sets

\[ \text{Ker}(f) = \{(a, b) \in X \times X : f(a) = f(b)\}, \]

\[ (q =) \text{Coker}(f) = \{(a, b) \in X \times X : f(a) \neq f(b)\} \]

are compatible equality and coequality relations on \( X \) and we can construct the factor-set \( X/q \).

The following theorem is the main result in this paper:

**Theorem 4:** Let \((X, =_X, \neq_X, \alpha)\) and \((Y, =_Y, \neq_Y, \beta)\) be ordered sets under antiorders, where the apartness \( \neq_Y \) is tight. If \( \varphi : X \rightarrow Y \) is reverse isotone strongly extensional function, then there exists a strongly extensional and embedding reverse isotone bijection

\[ ((X, =_X, \neq_X, \alpha, c(R))/q, =_{1, \neq_{1}}, \gamma) \rightarrow (\text{Im}(\varphi), =_Y , \neq_Y, \beta) \]

where \( c(R) \) is the biggest quasi-antiorder relation on \( X \) under \( R = \alpha \cap \text{Coker}(\varphi) \),

\[ q = c(R) \cup c(R)^{-1} \]

and \( \gamma \) is the antiorder induced by the quasi-antiorder \( c(R) \).

If the condition \( \alpha \cap \alpha^{-1} = \emptyset \) holds, then the there exists the isomorphism

\[ (X, =_X, \neq_X, \alpha, c(R))/q, =_{1, \neq_{1}}, \gamma) \cong (\text{Im}(\varphi), =_Y , \neq_Y, \beta). \]

**Proof:**

(1) Let \((X, =_X, \cong_X, \alpha)\) and \((Y, =_Y, \neq_Y, \beta)\) be ordered sets under antiorders \( \alpha \) and \( \beta \) respectively, and \( \varphi : X \rightarrow Y \) a strongly extensional mapping. Then the relation \( \varphi^{-1}(\beta) = \{(a, b) \in X \times X : (\varphi(a), \varphi(b)) \in \beta\} \) is a quasi-antiorder on \( X \), the relation \( \text{Coker} = \{(a, b) \in X \times X : \varphi(a) \neq_Y \varphi(b)\} \) is coequality relation on \( X \) compatible with equality relation \( \text{Ker} \varphi = \varphi^{-1} \circ \varphi \), and \( \text{Coker} \varphi \supseteq \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1} \) holds. Also, since the relation \( \beta \) is linear we have \( \text{Coker} \varphi = \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1} \). Indeed, \( \alpha \subseteq \neq \) and \( \varphi^{-1}(\beta) \subseteq \alpha \).

Since the relation \( \beta \) is linear, we have

\[ (a, b) \in \text{Coker} \varphi \iff \varphi(a) \neq_Y \varphi(b) \]

\[ \Rightarrow (\varphi(a), \varphi(b)) \in \beta \]

\[ \Leftarrow (a, b) \in \varphi^{-1}(\beta) \vee (b, a) \in \varphi^{-1}(\beta). \]

At the other side, if \( (a, b) \in \varphi^{-1}(\beta) \) or \( (b, a) \in \varphi^{-1}(\beta) \), then \( (\varphi(a), \varphi(b)) \in \beta \).
\( \beta \not\in \neq_Y \) or \((\varphi(b), \varphi(a)) \in \beta \subseteq \neq_Y \). Therefore, \( Coker(\varphi) = \varphi^{-1}(\beta)(\varphi^{-1}(\beta))^{-1} \).

(2) It is easily to conclude that \( Coker(\varphi) \subseteq \alpha \cup \alpha^{-1} \).

(3) The family \( A_R \) of quasi-antiorder relations on \( X \) under relation \( R = \alpha \cap Coker(\varphi) \) is not empty, because \( \varphi^{-1}(\beta) \subseteq R \). Then, by Theorem 3, there exists the biggest quasi-antiorder relation \( c(R) \) under \( R \). Put \( q = c(R) \cup (c(R))^{-1} \).

We can construct, according Lemma 0, the ordered factor-set \(((X, =_X, \neq_X, c(R))/q, =_1, \neq_1)\) under antiorder relation \( \gamma \) on \( X/q \), defined by \((aq, bq) \in \gamma\) if and only if \((a, b) \in c(R)\).

(4) We wish to show that \( Coker(\varphi) = q = c(R) \cup (c(R))^{-1} \). The first, by definition of \( c(R) \), \( c(R) \) is the biggest quasi-antiorder relation under \( R \). So, we have \( c(R) \subseteq Coker(\varphi) \) and \((c(R))^{-1} \subseteq (Coker(\varphi))^{-1} = Coker(\varphi) \) because the relation \( Coker(\varphi) \) is symmetric. Therefore, holds \( c(R) \cup (c(R))^{-1} \subseteq Coker(\varphi) \).

The second, the relation \( \varphi^{-1}(\beta) \) is a quasi-antiorder under \( R = \alpha \cap Coker(\varphi) \). So, it must be \( \varphi^{-1}(\beta) \subseteq c(R) \) because the relation \( c(R) \) is the biggest under \( R \). Thus, it must be \((\varphi^{-1}(\beta))^{-1} \subseteq (c(R))^{-1} \). Therefore, it must be \( Coker(\varphi) = \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1} \subseteq c(R) \cup (c(R))^{-1} \).

If \( \alpha \cap \alpha^{-1} = \emptyset \) holds, then easy to verify that \( c(R) \cap (c(R))^{-1} = \emptyset \) holds too.

(5) By Lemma 0, the set \(((X, =_X, \neq_X, =_1, \neq_1)\) is ordered set under the antiorder \( \gamma \) on \( X/q \) defined by

\[
(aq, bq) \in \gamma \iff (a, b) \in c(R).
\]

If \( \alpha \cap \alpha^{-1} = \emptyset \), then \( c(R) \cap (c(R))^{-1} = \emptyset \), because \( c(R) \cup (c(R))^{-1} \subseteq \alpha \cup \alpha^{-1} \). It remains to construct mapping \( \phi: X/q \rightarrow Im(\varphi) (\subseteq Y) \). Define \( \phi(aq) = \varphi(a) \) for any \( a \) in \( X \).

(a) This mapping is well defined because if \( aq =_1 bq \), i.e. \((a, b) \triangleright q = c(R) \cup (c(R))^{-1} = Coker(\varphi) \), then \( -(\varphi(a) \neq_Y \varphi(b)) \) holds. Since the apartness \( \neq_Y \) is tight, it implies that \( \varphi(a) =_Y \varphi(b) \), i.e. \( \phi(aq) =_Y \phi(bq) \).

(b) Suppose that \( \phi(aq) \neq_Y \phi(bq) \), i.e. suppose that \( \varphi(a) \neq_Y \varphi(b) \), i.e. suppose that \((a, b) \in Coker(\varphi) \). Then \( aq \neq_1 bq \). Therefore, the mapping is strongly extensional function from \( X/q \) into \( Y \).

(c) If \( y \in Im(\varphi) \), then for some \( x \in X \), \( \phi(xq) =_Y \varphi(x) =_Y y \). Thus, the mapping \( \phi: X/q \rightarrow Im(\varphi) \) is a strongly extensional and surjective function.

(d) If \( \phi(aq) =_Y \phi(bq) \), then \( \varphi(a) =_Y \varphi(b) \). Let \((u, v) \) be an arbitrary element of \( Coker(\varphi) \).

Then from \( \varphi(u) \neq_Y \varphi(v) \) follows

\[
\varphi(u) \neq_Y \varphi(a) \varphi(b) \neq_Y \varphi(v).
\]

Since \( \varphi(a) \neq_Y \varphi(b) \) is impossible, we conclude that above disjunction follows

\[
\varphi(u) \neq_Y \varphi(a) \varphi(b) \neq_Y \varphi(v).
\]

and \( u \neq X a \) or \( b \neq X v \). So, \((u, v) \neq X \times Y \). This means \( (a, b) \triangleright Coker(\varphi) \).

Therefore \( aq \neq_1 bq \). Hence, the mapping \( \phi \) is an injective function.

(e) Now, let be \( aq \neq_1 bq \). Then \((a, b) \in Coker(\varphi) \), i.e. \( \varphi(a) \neq_Y \varphi(b) \).
Therefore, in this case, we have $\phi(aq) \neq_Y \phi(bq)$. So, the function $\phi$ is an embedding.

(f) The first, we wish to prove that the function $\phi$ is reverse isotone bijection. If $(\phi(aq), \phi(bq)) \in \beta$, i.e. if $(\varphi(a), \varphi(b)) \in \beta (\subseteq \neq_Y)$, then $(a, b) \in \varphi^{-1}(\beta) \subseteq c(R)$ by the second part of the point (3) of this proof. Therefore, $(aq, bq) \in \gamma$. So, the bijection is reverse isotone.

The second, we wish to prove that the function $\phi$ is isotone bijection. Let $(aq, bq) \in \gamma (\subseteq \neq_1)$, i.e. let $(a, b) \in c(R) (\subseteq \alpha)$. Since the function $\phi$ is an embedding, then $\phi(aq) \neq_Y \phi(bq)$. So, must be $(\phi(aq), \phi(bq)) \in \beta$ or $(\phi(bq), \phi(aq)) \in \beta$. Suppose that $(\phi(bq), \phi(aq)) \in \beta$, i.e. suppose that $(\varphi(b), \varphi(a)) \in \beta$ holds. Thus we conclude that $(b, a) \in \alpha$ because the function $\varphi$ is reverse isotone mapping. If the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, then the case $(\phi(bq), \phi(aq)) \in \beta$ is impossible. Now, we have to have $(\phi(aq), \phi(bq)) \in \beta$. So, in the case that the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, the mapping $\phi$ is isotone.

At end of this conclusion we have that there exists strongly extensional and embedding reverse isotone bijection from $((X, \!=\!, \neq, c(R))/q, \!=\!, \neq_1)$ onto $(\text{Im}(\varphi), \!=\!, \neq_Y, \neq_1, \gamma)$. If the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, then there exists the isomorphism $((X, \!=\!, \neq, c(R))/q, \!=\!, \neq_1, \gamma) \cong (\text{Im}(\varphi), \!=\!, \neq_Y, \neq_1, \gamma)$.

\begin{itemize}
  \item Note. Let $(X, \!=\!, \neq, \alpha), (Y, \!=\!, \neq, \beta)$ be ordered sets under antiorders $\alpha$ and $\beta$ respective, and let $\varphi : X \rightarrow Y$ be a strongly extensional mapping from $X$ into $Y$. Then, by point (1) in the proof of the Theorem 4, the relation induced there $\varphi^{-1}(\beta)$ is quasi-antiorder relation on $X$. Then:
  \begin{enumerate}
    \item $\varphi$ is isotone if $\alpha \subseteq \varphi^{-1}(\beta)$;
    \item $\varphi$ is reverse isotone if and only if $\varphi^{-1}(\beta) \subseteq \alpha$.
  \end{enumerate}
\end{itemize}

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